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Observer-based H_∞ consensus for linear multi-agent systems subject to measurement outliers

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Abstract

The consensus problem has a crucial role in theoretical and practical aspects of interconnected systems, for instance in smart computing field. This paper is concerned with the observer-based H_∞ consensus problem for a class of linear multi-agent systems subject to measurement outliers. In the addressed observer-based H_∞ consensus problem, measurement outliers can affect the estimation accuracy and thus affect the consensus performance. To assuage the impact of measurement outliers, a control protocol based on an observer containing a saturation function with variable saturation limits is proposed. The purpose of this paper is to find a solution to the addressed H_∞ consensus problem for a class of linear multi-agent systems subject to measurement outliers by designing an observer-based control protocol such that multi-agent systems can fulfill the H_∞ consensus performance over a finite horizon. With the aid of Lyapunov theory, the sufficient condition is established to guarantee that the consensus error dynamic system satisfies the H_∞ consensus performance. Then, the linear matrix inequality (LMI) approach is used to obtain the desired parameters of the observer-based control protocol. Finally, the effectiveness of the proposed control protocol is verified in a simulation environment.

Aim: The purpose of this study is to find a solution to the addressed H_∞ consensus problem for a class of linear multi-agent systems subject to measurement outliers.

Methods: By choosing a suitable Lyapunov function, the sufficient condition is obtained, which can guarantee the consensus error dynamic system satisfying the given H_∞ consensus performance. The LMI approach is employed to design the desired controller. With Matlab LMI toolbox, a numerical example was conducted to demonstrate the effectiveness of the proposed observer-based control protocol in the simulation environment.

Results: A satisfactory consensus performance can be guaranteed for multi-agent systems subject to measurement outliers under the proposed observer-based control protocol. The constructed saturation function with variable saturation limits can mitigate the effect of measurement outliers by dynamically regulating saturation limits. Compared with the traditional observer-based control protocol, in this paper, the proposed observer-based control protocol shows robustness against measurement outliers.

Conclusion: A solution to the addressed multi-agent consensus problem subject to measurement outliers was found by designing an observer-based consensus controller. The obtained results can be extended to sensor networks, neural networks, and nonlinear multi-agent systems.

Keywords: Multi-agent systems, measurement outliers, H_∞ consensus, linear matrix inequality, saturation with variable saturation limits

INTRODUCTION

A smart computing system is usually composed of a number of smart mobile nodes. To maintain the prespecific system performance requirement in the presence of certain sudden cases, researchers have employed many intelligent methods to cope with the computing problem for different types of practical systems. With the development of science and technology, the study of the smart computing problem for interconnected systems has become a hot research topic, and much effort has been made in many application fields, such as smart grids[1]. It is worth mentioning that the consensus problem has shown a crucial role in theoretical and practical aspects of interconnected systems and has stirred up a wave of research (see[2,3]).

In the past few decades, the research on collective behaviors of multiple networked agents has attracted attention owing to the wide range of practical applications referring to robotics, satellite networks, unmanned vehicles, sensor networks, and so on. Consensus, as one of the most typical collective behaviors, aims to let the states of a group of homogeneous or heterogeneous agents in light of the local shared neighboring information reach a common value[4,5]. Focusing on the consensus control problem, researchers in different research fields have made great efforts and numerous research results have been published (see[6] and the references therein). To cater to the robustness requirement of multi-agent systems, the H_∞ consensus problem is put forward in ref.[7], which is extended to the H_∞ consensus control problem in ref.[8]. Until now, the H_∞ consensus problem has been studied deeply and many remarkable results have been published (see, e.g.,[9–16]). In particular, the H_∞-consensus control problem under round-robin protocol is studied for a class of linear multi-agent systems in ref.[15]. Further, in ref.[16], the leader–follower H_∞-consensus problem is investigated for a class of nonlinear multi-agent systems under semi-Markovian switching topologies with partially unknown transition rates.

In light of the given network topology, how to design an efficient H_∞ consensus control protocol is a pivotal issue for the individual agent[17]. Under the state-feedback-based control protocol, one agent in multi-agent systems, by using its own state information and the state information shared by its neighboring agents, can make its state converge to a given common state. Because of the simpleness and convenience of design, the state-feedback-based consensus control problem has obtained widely attention[6,18]. Unfortunately, in practical engineering, it is sometimes difficult to attain the full state information. Instead, observer-type distributed consensus control protocols, as one type of the measurement-based control protocols, have been developed.

In recent decades, the observer-based H_∞ consensus problem for multi-agent systems has attracted a large amount of research attention, and many remarkable research results have been published in the literature (see, e.g.[4,19–21]). In ref.[19], a robust yet reliable consensus control law against actuator faults is proposed, under which the desired control performance can be ensured. In ref.[4], the observer-type event-triggered H_∞ consensus control problem is studied in a finite-time horizon for a class of discrete time-varying multi-agent systems subject to external disturbances and missing measurements.

In the above-mentioned studies, the impacts of missing measurements, actuator faults, and redundant channels on multi-agent consensus performance have been investigated. In practical applications, it is noted that outlier is another common phenomenon mainly originating from sensor or communication failures, disturbances in the environment, human errors, and so on, which will contaminate the measurements and thus affect the control performance. To mitigate the effect of measurement outliers, several research efforts have been made (see[22–26]). For example, in[22], a novel observer including a saturation function with variable saturation limits is constructed. Based on the results, the state estimation problem for neural networks is studied in ref.[23] and the fault detection problem is considered in T-S fuzzy system in ref.[26]. In addition, the method proposed for outliers detection in wireless sensor networks has been studied, which is implemented within a hierarchical multi-agent framework in ref.[27]. It is worth mentioning that the H_∞ consensus control problem based on such a novel observer for linear multi-agent systems has not yet gained adequate attention, and this constitutes the main motivation of this paper.

Based on above discussions, in this paper, the observer-based H_∞ consensus problem is studied for a class of linear multi-agent systems subject to measurement outliers. The main contributions of this paper are summarized as follows: *(1) measurement outliers are considered for the first time for linear multi-agent consensus control problem; (2) a saturation function with variable saturation limits is introduced to the observer;* and (3) a novel observer-based control protocol is proposed, under which linear multi-agent systems subject to measurement outliers can achieve H_∞ consensus performance over a finite horizon.

PROBLEM FORMULATION

Consider a multi-agent system with N agents under a fixed network topology. Let $\hat{\mathcal{G}} = \{\hat{\mathcal{V}}, \hat{\mathcal{E}}\}$ be a directed graph including a node set $\hat{\mathcal{V}} = \{1, 2, \dots, N\}$ and an edge set $\hat{\mathcal{E}} \subseteq \hat{\mathcal{V}} \times \hat{\mathcal{V}}$. If an edge exists between nodes i and j, we denote $(i, j) \in \hat{\mathcal{E}}$ and node j is called a neighbor of node i and vice versa; if no edge exists between nodes i and j, it means $(i, j) \notin \hat{\mathcal{E}}$. Let $\mathcal{N}_i$ denote all neighbors of node i. Moreover, define the Laplacian matrix $\mathcal{L} = [l_{ij}]_{N\times N}$ with $l_{ij} = -1$ if $(i, j) \in \hat{\mathcal{E}}$, $l_{ij} = 0$ if $(i, j) \notin \hat{\mathcal{E}}$, and $l_{ii} = -\sum_{j=1}^{N} l_{ij}$.

Consider a discrete linear multi-agent system with N agents. The dynamics of ith agent can be expressed by

$$x_i(k+1) = Ax_i(k) + Bu_i(k) + D\varpi_i(k) \tag{1}$$

$$y_i(k) = Cx_i(k) + Ev_i(k) \tag{2}$$

where $x_i(k) \in \mathbb{R}^{n_x}$ is the state, $u_i(k) \in \mathbb{R}^{n_u}$ is the control input, $y_i(k) \in \mathbb{R}^{n_y}$ is the measurement output, and $\varpi_i(k) \in \mathbb{R}^{n_\varpi}$ and $v_i(k) \in \mathbb{R}^{n_v}$ are the system noise and measurement disturbance, respectively, which are assumed to belong to $l_2[0, M]$. A, B, C, D, and E are known time-invariant matrices with appropriate dimensions.

In this paper, we consider a case that outliers may occur in the measurement. In practical applications, outliers may originate from sensor or communication failures, disturbances in the environment, human errors, and so on. To mitigate the effect of measurement outliers, we construct an observer-based control protocol of the following form for agent i:

$$\begin{cases} \hat{x}_i(k+1) & = A\hat{x}_i(k) + Bu_i(k) + G\mathrm{sat}_{\sigma_i(k)}(s_i(k)) \\ \sigma_i(k) & = \sqrt{\frac{\bar{\sigma}_i(k)}{w_{i,h}}} \\ \bar{\sigma}_i(k+1) & = \lambda_i\bar{\sigma}_i(k) + s_i^T(k)R_is_i(k) \\ s_i(k) & = y_i(k) - C\hat{x}_i(k) \\ u_i(k) & = K\sum_{j\in\mathcal{N}_i}(\hat{x}_j(k) - \hat{x}_i(k)) \end{cases} \tag{3}$$

where $\hat{x}_i(k)$ is the state of the observer i, $\sigma_i(k) = [\sigma_{i,1}(k), \sigma_{i,2}(k), \ldots, \sigma_{i,h}(k)]^T$ $(h = 1, 2, \ldots, n_y)$ denotes the variable non-negative saturation limit vector, $s_i(k) = [s_{i,1}(k), s_{i,2}(k), \ldots, s_{i,h}(k)]^T$ is the innovation vector, and $\mathrm{sat}_{\sigma_i(k)}(s_i(k))$ is a saturation function of the following form:

$$\mathrm{sat}_{\sigma_i(k)}(s_i(k)) = \begin{bmatrix} \mathrm{sat}_{\sigma_{i,1}(k)}(s_{i,1}(k)) \\ \mathrm{sat}_{\sigma_{i,2}(k)}(s_{i,2}(k)) \\ \vdots \\ \mathrm{sat}_{\sigma_{i,n_y}(k)}(s_{i,n_y}(k)) \end{bmatrix}$$

with $\mathrm{sat}_{\sigma_{i,h}(k)}(s_{i,h}(k)) = \max\{-\sigma_{i,h}(k), \min\{\sigma_{i,h}(k), s_{i,h}(k)\}\}$ being a saturation function. The scalars $\lambda_i \in [0, 1)$, $w_{i,h} > 0$, matrices $R_i = R_i^T > 0$, and G and K are controller parameters to be designed below. Define $W_i = \mathrm{diag}(w_{i,1}, w_{i,2}, \ldots, w_{i,n_y})$.

An auxiliary variable $q_i(k)$ is introduced to simplify the notation expression, which is defined as follows:

$$q_i(k) = s_i(k) - \mathrm{sat}_{\sigma_i(k)}(s_i(k)). \tag{4}$$

Denote

$$\begin{aligned} &\mathcal{A} = \mathrm{diag}_N\{A\}, \mathcal{B} = \mathrm{diag}_N\{B\}, \mathcal{C} = \mathrm{diag}_N\{C\} \\ &\mathcal{D} = \mathrm{diag}_N\{D\}, \mathcal{E} = \mathrm{diag}_N\{E\}, \mathcal{G} = \mathrm{diag}_N\{G\} \\ &W = \mathrm{diag}_N\{W_i\}, R = \mathrm{diag}_N\{R_i\}, \lambda = \mathrm{diag}_N\{\lambda_i\} \\ &x(k) = [x_1^T(k), \ldots, x_N^T(k)]^T, y(k) = [y_1^T(k), \ldots, y_N^T(k)]^T \\ &\hat{x}(k) = [\hat{x}_1^T(k), \ldots, \hat{x}_N^T(k)]^T, \varpi(k) = [\varpi_1^T(k), \ldots, \varpi_N^T(k)]^T \\ &v(k) = [v_1^T(k), \ldots, v_N^T(k)]^T, q(k) = [q_1^T(k), \ldots, q_N^T(k)]^T \\ &\bar{\sigma}(k) = [\bar{\sigma}_1^T(k), \ldots, \bar{\sigma}_N^T(k)]^T, u(k) = [u_1^T(k), \ldots, u_N^T(k)]^T \\ &s(k) = [s_1^T(k), \ldots, s_N^T(k)]^T, \mathbf{1}_N = [\underbrace{1, 1, \ldots, 1}_{N}]^T \\ &\mathrm{sat}_\sigma^s(k) = [(\mathrm{sat}_{\sigma_1(k)}(s_1(k)))^T, \ldots, (\mathrm{sat}_{\sigma_N(k)}(s_N(k)))^T]^T. \end{aligned}$$

Then, Equations (1) and (2) can be rewritten as follows:

$$x(k+1) = \mathcal{A}x(k) + \mathcal{B}u(k) + \mathcal{D}\varpi(k) \tag{5}$$

$$y(k) = \mathcal{C}x(k) + \mathcal{E}v(k). \tag{6}$$

The proposed observer-based control protocol in Equation (3) can be rewritten as follows:

$$\begin{cases} \hat{x}(k+1) & = \mathcal{A}\hat{x}(k) + \mathcal{B}u(k) + \mathcal{G}\mathrm{sat}_\sigma^s(k) \\ \mathbf{1}_N^T\bar{\sigma}(k+1) & = \mathbf{1}_N^T\lambda\bar{\sigma}(k) + s^T(k)Rs(k) \\ s(k) & = y(k) - \mathcal{C}\hat{x}(k) \\ u(k) & = K\otimes\mathcal{L}\hat{x}(k) \end{cases} \tag{7}$$

Then, the augmented auxiliary variable $q(k)$ has the following form:

$$q(k) = s(k) - \text{sat}_{\sigma}^{s}(k). \tag{8}$$

By defining $e(k) = x(k) - \hat{x}(k)$ and combining Equations (5)–(8), the following error dynamic system can be obtained:

$$e(k+1) = (\mathcal{A} - \mathcal{G}C)e(k) + \mathcal{D}\varpi(k) - \mathcal{G}v(k) + \mathcal{G}q(k). \tag{9}$$

Denote

$$\begin{aligned} \eta(k) =& [x^T(k)\ e^T(k)]^T, \mathscr{D} = [\mathcal{D}^T\ \mathcal{D}^T]^T, \mathscr{G} = [0\ \mathcal{G}^T]^T \\ \mathscr{A} =& \begin{bmatrix} \mathcal{A} + \mathcal{B}K \otimes \mathcal{L} & -\mathcal{B}K \otimes \mathcal{L} \\ 0 & \mathcal{A} - \mathcal{G}C \end{bmatrix}. \end{aligned}$$

Then, it follows from Equations (5) and (9) that

$$\eta(k+1) = \mathscr{A}\eta(k) + \mathscr{D}\varpi(k) - \mathscr{G}v(k) + \mathscr{G}q(k). \tag{10}$$

Definition 1 *Given the positive definite matrix $U = U^T > 0$ and the disturbance attenuation level $\gamma > 0$, if the following inequality*

$$\|\eta(k) - \ell\eta(k)\|_{[0,M]}^2 < \gamma^2\left\{\|\varpi(k)\|_{[0,M]}^2 + \|v(k)\|_{[0,M]}^2 + \eta^T(0)U\eta(0)\right\} \tag{11}$$

holds, where $\ell = (\frac{1}{N}\mathbf{1}_N\mathbf{1}_N^T) \otimes I_{2n_x}$, it is said that the multi-agent system in Equation (1) can achieve the H_∞ consensus performance index in Equation (11) over a finite horizon $[0, M]$.

In this paper, our objective is to design appropriate parameter matrices and scalars $\{G, \lambda_i, R_i, W_i, K\}$ for the proposed observer-based control protocol in Equation (3) such that the multi-agent system in Equation (1) can achieve consensus.

MAIN RESULTS

In this section, a solution is found to the addressed multi-agent consensus problem by properly designing the desired controller in Equation (3). The following theorem gives a sufficient condition under which the augmented system in Equation (10) can achieve the given H_∞ consensus performance in Equation (11).

Firstly, we give the following assumption and lemma which are useful to obtain our main results.

Assumption 1 *The matrix B is a full-column rank matrix, i.e.,* $\text{rank}(B) = c_B$.

For the matrix B, there exist two orthogonal matrices $T \in \mathbb{R}^{l\times l}$ and $Z \in \mathbb{R}^{c_B\times c_B}$, such that

$$\hat{B} = TBZ = \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} BZ = \begin{bmatrix} \Lambda \\ 0 \end{bmatrix} \tag{12}$$

where $T_1 \in \mathbb{R}^{c_B\times l}$, $T_2 \in \mathbb{R}^{(l-c_B)\times l}$ and $\Lambda = \text{diag}\{\Lambda_1, \Lambda_2, \ldots, \Lambda_{c_B}\}$ with Λ_j $(j = 1, 2, \ldots, c_B)$ being nonzero singular values of B.

Lemma 1 *Assume the matrix $B \in \mathbb{R}^{l\times c_B}$ is a full-column rank matrix. If matrix P_1 has the structure*

$$P_1 = T^T \begin{bmatrix} P_{11} & 0 \\ 0 & P_{22} \end{bmatrix} T = T_1^T P_{11} T_1 + T_2^T P_{22} T_2 \tag{13}$$

where $P_{11} \in \mathbb{R}^{c_B\times c_B} > 0$, $P_{22} \in \mathbb{R}^{(l-c_B)\times(l-c_B)} > 0$, and the definitions of T_1 and T_2 are given in Equation (12), then a nonsingular matrix $P_0 \in \mathbb{R}^{c_B\times c_B}$ satisfies $BP_0 = P_1 B$.

Theorem 1 *Consider the linear multi-agent system in Equation (1) and let the disturbance attenuation level γ, a positive definite matrix U, and controller parameters $\{G, K\}$ be given. The augmented system in Equation (10) can achieve the H_∞ consensus performance in Equation (11) if there exist a positive definite matrix P and some controller parameter matrices and scalars $\{\lambda_i, R_i, W_i\}$, under the initial condition $\eta^T(0)P\eta(0) - \mathbf{1}_N^T\bar{\sigma}(0) - \gamma^2\eta^T(0)U\eta(0) \le 0$, satisfying the following LMI:*

$$\mathcal{H} + \mathcal{Y}^T P \mathcal{Y} < 0 \tag{14}$$

where

$$\mathcal{H} = \begin{bmatrix} \mathcal{H}_1 & 0 & F^TC^TR & F^TC^TW & 0 \\ * & -\gamma^2 I & 0 & 0 & 0 \\ * & * & -\gamma^2 I + R & W & 0 \\ * & * & * & -2W & 0 \\ * & * & * & * & \mathcal{H}_4 \end{bmatrix}$$

$$\mathcal{H}_1 = -P + \tilde{\ell} + F^TC^TRCF, \mathcal{H}_4 = \mathrm{diag}(\mathbf{1}_N^T\lambda - \mathbf{1}_N^T)$$

$$\mathcal{Y} = [\mathscr{A}\ \mathscr{D}\ -\mathscr{G}\ \mathscr{G}\ 0], \tilde{\ell} = I_N \otimes I_{2n_x} - (\frac{1}{N}\mathbf{1}_N\mathbf{1}_N^T) \otimes I_{2n_x}. \tag{15}$$

proof 1 *Construct the following Lyapunov-like functional function for the augmented system in Equation (10):*

$$V(k) = \eta^T(k)P\eta(k) + \mathbf{1}_N^T\bar{\sigma}(k).$$

Then, one has

$$\begin{aligned}\Delta V(k) =& \eta^T(k+1)P\eta(k+1) + \mathbf{1}_N^T\bar{\sigma}(k+1) - \eta^T(k)P\eta(k) - \mathbf{1}_N^T\bar{\sigma}(k) \\ =& \big(\mathscr{A}\eta(k) + \mathscr{D}\varpi(k) - \mathscr{G}v(k) + \mathscr{G}q(k)\big)^T P\big(\mathscr{A}\eta(k) + \mathscr{D}\varpi(k) - \mathscr{G}v(k) + \mathscr{G}q(k)\big) \\ &+ \mathbf{1}_N^T\lambda\bar{\sigma}(k) + (y(k) - C\hat{x}(k))^T R(y(k) - C\hat{x}(k)) - \eta^T(k)P\eta(k) - \mathbf{1}_N^T\bar{\sigma}(k).\end{aligned} \tag{16}$$

According to Lemma 1.6 in[28]*, the following sector constraint condition can be obtained:*

$$2q^T(k)W(CF\eta(k) + v(k) - q(k)) \ge 0 \tag{17}$$

where $F = [0\ I]$.

Taking the H_∞ consensus performance in Equation (11) into consideration, it follows from Equations (16) and (17) that

$$\begin{aligned}&\|\eta(k) - \ell\eta(k)\|^2_{[0,M]} - \gamma^2\|\varpi(k)\|^2_{[0,M]} - \gamma^2\|v(k)\|^2_{[0,M]} - \gamma^2\eta^T(0)U\eta(0) \\ \le& \sum_{k=0}^{M}\{\|\eta(k) - \ell\eta(k)\|^2 - \gamma^2\|\varpi(k)\|^2 - \gamma^2\|v(k)\|^2\} \\ &+ \sum_{k=0}^{M}\{\eta^T(k+1)P\eta(k+1) + \mathbf{1}_N^T\bar{\sigma}(k+1) - \eta^T(k)P\eta(k) - \mathbf{1}_N^T\bar{\sigma}(k) \\ &+ 2q^T(k)W(CF\eta(k) + v(k) - q(k))\} + \eta^T(0)P\eta(0) - \mathbf{1}_N^T\bar{\sigma}(0) - \gamma^2\eta^T(0)U\eta(0) \\ =& \sum_{k=0}^{M}\{\Gamma^T(k)(\mathcal{H} + \mathcal{Y}^T P\mathcal{Y})\Gamma(k)\} + \eta^T(0)P\eta(0) - \mathbf{1}_N^T\bar{\sigma}(0) - \gamma^2\eta^T(0)U\eta(0)\end{aligned} \tag{18}$$

where $\Gamma(k) = [\eta^T(k)\ \varpi^T(k)\ v^T(k)\ q^T(k)\ (\bar{\sigma}^{\frac{1}{2}}(k))^T]^T$ *and* $\mathcal{H}$ *and* $\mathcal{Y}$ *are given in Equation (15).*

By noticing the initial condition $\eta^T(0)P\eta(0) - \mathbf{1}_N^T\bar{\sigma}(0) - \gamma^2\eta^T(0)U\eta(0) \le 0$, *we have from Equation (18) that the* H_∞ *consensus performance requirement in Equation (11) is achieved when Equation (14) holds. The proof is now complete.*

According to the results in Theorem 1, the sufficient condition guaranteeing the augmented system in Equation (10) achieves the H_∞ consensus performance in Equation (11) is obtained. Thus, we are now ready to design the parameters of the proposed controller in Equation (3) in terms of the following theorem.

Theorem 2 *Consider the linear multi-agent system in Equation (1) and let the disturbance attenuation level* γ *and a positive definite matrix* U *be given. The augmented system in Equation (10) can achieve the* H_∞ *consensus performance in Equation (11) if there exist a positive definite matrix* $P = \operatorname{diag}\{P_1, P_2\}$ *with* $P_1 = T_1^T P_{11} T_1 + T_2^T P_{22} T_2$, $P_{11} \in \mathbb{R}^{c_B \times c_B} > 0$, $P_{22} \in \mathbb{R}^{(l-c_B)\times(l-c_B)} > 0$, T_1 *and* T_2 *being given in Equation (12), and some parameter matrices and scalars* $\{G, \lambda_i, R_i, W_i, K_0\}$, *under the initial condition* $\eta^T(0)P\eta(0) - \mathbf{1}_N^T\bar{\sigma}(0) - \gamma^2\eta^T(0)U\eta(0) \le 0$, *satisfying the following LMI:*

$$\begin{bmatrix} \mathcal{H} & \mathcal{Y}_0^T \\ * & -P \end{bmatrix} < 0 \tag{19}$$

where

$$\mathcal{Y}_0 = [\mathscr{A}_0\ P\mathscr{D}\ -\mathscr{G}_0\ \mathscr{G}_0\ 0], \mathscr{G}_0 = [0\ \mathcal{G}_0^T]^T$$
$$\mathscr{A}_0 = \begin{bmatrix} P_1\mathcal{A} + \mathcal{B}K_0 \otimes \mathcal{L} & -\mathcal{B}K_0 \otimes \mathcal{L} \\ 0 & P_2\mathcal{A} - \mathcal{G}_0 C \end{bmatrix}$$

$\mathcal{H}$ *is given in Equation (15).*

Moreover, the parameters of desired controller are given by

$$K = Z\Lambda^{-1}P_{11}^{-1}\Lambda Z^T K_0, \quad \mathcal{G} = P_2^{-1}\mathcal{G}_0. \tag{20}$$

proof 2 *Noting Assumption 1 and Lemma 1, we can obtain that there exists a nonsingular matrix* $P_0 \in \mathbb{R}^{c_m \times c_m}$ *that satisfies* $BP_0 = P_1 B$. *Then, we can obtain* P_0 *as follows:*

$$P_0 = (Z^T)^{-1}\Lambda^{-1}P_{11}\Lambda Z^T. \tag{21}$$

Let $P_1 B = BP_0$, $K_0 = P_0 K$ *and* $\mathcal{G}_0 = P_2\mathcal{G}$; *then, in light of the Schur complement lemma, Equation (19) can easily be obtained from Equation (14). The proof of this theorem is now complete.*

In Equation (3), by letting $\sigma_i(k) = \sigma \to \infty$ $(i = 1, 2, \ldots, N)$, the following general control protocol can be obtained for agent i:

$$\begin{cases} \hat{\bar{x}}_i(k+1) & = A\hat{\bar{x}}_i(k) + B\bar{u}_i(k) + \bar{G}\bar{s}_i(k) \\ \bar{s}_i(k) & = y_i(k) - C\hat{\bar{x}}_i(k) \\ \bar{u}_i(k) & = \bar{K}\sum_{j\in N_i}(\hat{\bar{x}}_j(k) - \hat{\bar{x}}_i(k)) \end{cases} \tag{22}$$

By defining $\bar{e}(k) = x(k) - \hat{\bar{x}}(k)$ and combining Equations (5), (5) and (22), we can obtain

$$\bar{e}(k+1) = (\mathcal{A} - \bar{\mathcal{G}}C)e(k) + \mathcal{D}\varpi(k) - \bar{\mathcal{G}}v(k).$$

Denote

$$\bar{\eta}(k)=\begin{bmatrix}x(k)\\ \bar{e}(k)\end{bmatrix},\bar{\mathscr{G}}=\begin{bmatrix}0\\ \bar{\mathcal{G}}\end{bmatrix},\bar{\mathscr{A}}=\begin{bmatrix}\mathcal{A}+\mathcal{B}\bar{K}\otimes\mathcal{L} & -\mathcal{B}\bar{K}\otimes\mathcal{L}\\ 0 & \mathcal{A}-\bar{\mathcal{G}}C\end{bmatrix}.$$

Then, we obtain

$$\bar{\eta}(k+1)=\bar{\mathscr{A}}\bar{\eta}(k)+\mathscr{D}\varpi(k)-\bar{\mathscr{G}}v(k). \tag{23}$$

The following corollary can guarantee that the augmented system in Equation (23) satisfies the following H_∞ consensus performance requirement

$$\|\bar{\eta}(k)-\ell\bar{\eta}(k)\|^2_{[0,M]}<\bar{\gamma}^2\{\|\varpi(k)\|^2_{[0,M]}+\|v(k)\|^2_{[0,M]}+\bar{\eta}^T(0)\bar{U}\bar{\eta}(0)\} \tag{24}$$

where $\bar{U}=\bar{U}^T>0$ is a positive definite matrix and $\bar{\gamma}>0$ is the disturbance attenuation level, which means that the consensus performance of multi-agent system in Equation (1) can be guaranteed.

Corollary 1 *Consider the multi-agent system Equation (1) and let the disturbance attenuation level $\bar{\gamma}$ and a positive definite matrix $\bar{U}$ be given. The augmented system in Equation (23) achieves the H_∞ consensus performance requirement in Equation (24) if there exist a positive definite matrix $\bar{P}=\mathrm{diag}\{\bar{P}_1,\bar{P}_2\}$ with $\bar{P}_1=\bar{T}_1^T\bar{P}_{11}\bar{T}_1+\bar{T}_2^T\bar{P}_{22}\bar{T}_2$, $\bar{P}_{11}\in\mathbb{R}^{c_B\times c_B}>0$, $\bar{P}_{22}\in\mathbb{R}^{(l-c_B)\times(l-c_B)}>0$, $\bar{T}_1$ and $\bar{T}_2$ being given in Equation (12), and some parameter matrices $\{\bar{G},\bar{K}_0\}$, under the initial condition $\bar{P}-\bar{\gamma}^2\bar{U}\le 0$, satisfying the following LMI:*

$$\begin{bmatrix}-\bar{P}+\tilde{\ell} & 0 & 0 & \bar{\mathscr{A}}_0^T\\ * & -\gamma^2I & 0 & \bar{P}\mathscr{D}^T\\ * & * & -\gamma^2I & -\bar{\mathscr{G}}_0^T\\ * & * & * & -\bar{P}\end{bmatrix}<0. \tag{25}$$

Moreover, the parameters of desired controller are given by

$$\bar{K}=Z\Lambda^{-1}\bar{P}_{11}^{-1}\Lambda Z^T\bar{K}_0,\quad \bar{\mathcal{G}}=\bar{P}_2^{-1}\bar{\mathcal{G}}_0. \tag{26}$$

proof 3 *By letting $\sigma_i(k)=\sigma\to\infty$, $w_i(k)\to 0$, $R_i(k)=0$, and $\lambda_i(k)=0$ $(i=1,2,\ldots,N)$, the proof of this corollary can be obtained directly from Theorem 2.*

Remark 1 *The main contribution of this paper is that a novel observer containing a saturation function with variable saturation limits is introduced into the designed consensus control protocol. Because of the constraint of saturation function with variable saturation limits, the saturation limits can dynamically change when outliers occur. Based on such merit, the designed control protocol can mitigate the effect of measurement outliers effectively.*

Remark 2 *Thus far, a solution to the addressed H_∞ consensus problem for a class of linear multi-agent systems subject to measurement outliers has been found by designing an observer-based control law. The main results have been derived and shown in Equations (1) and (2). For the case of measurement without outliers, the corresponding result is given in Corollary 1. In the next section, a numerical example is provided in a simulation environment to verify the effectiveness of the proposed consensus control protocol.*

ILLUSTRATIVE EXAMPLE

In this section, a numerical example is provided to verify the effectiveness of the proposed control protocol in Equation (3).

Consider multi-agent systems including five agents. Given the Laplacian matrix as $\mathcal{L} = [2\ -1\ -1\ 0\ 0; -1\ 3\ -1\ -1\ 0; -1\ -1\ 4\ -1\ -1; 0\ -1\ -1\ 3\ -1; 0\ 0\ -1\ -1\ 2]$, the ith agent's state is $x_i(k) = [x_{i,1}(k)\ x_{i,2}(k)]^T$ and its dynamics is given in Equation (1) with

$$A = \begin{bmatrix} 1 & -0.5 \\ 0.2 & -1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, D = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, C = [1\ 1], E = 1.$$

The length of finite horizon M is set as $M = 100$. γ is chosen as $\gamma = 1.5$. $x_i(0)$ and $\hat{x}_i(0)$ $(i = 1, 2, \ldots, 5)$ obey the uniform distribution on the interval $[-2, 2]$. Set $\bar{\sigma}_i(0)$, $\varpi_i(k)$, and $v_i(k)$ as $\bar{\sigma}_i(0) = 1$, $\varpi_i(k) = e^{-10k}\sin(10k)$, and $v_i(k) = e^{-10k}\cos(10k)$ $(i = 1, 2, \ldots, 5)$, respectively.

The outliers considered in measurements are unit pulse signals, which enter the second agent at $k = 60$ and the fourth agent at $k = 80$. Next, by making a comparison between the results of Theorem 2 and Corollary 1, the effectiveness of the proposed control protocol in Equation (3) can be demonstrated. In terms of the results of Theorem 2 and Corollary 1, **by employing MATLAB LMI toolbox, the solutions of LMIs in Theorem 2 can be obtained as follows:**

$$\begin{aligned} & G = \begin{bmatrix} 0.1467 \\ -0.1165 \end{bmatrix}^T, \ K = \begin{bmatrix} 0.0373 \\ -0.0230 \end{bmatrix}^T, \\ & W_i = 0.7285,\ R_i = 0.2813,\ \lambda_i = 0.3948\ (i = 1, 2, \ldots, 5), \end{aligned}$$

and then the simulation results are shown in Figure 1–5, which depict the compared state trajectories of five agents, the evolution profiles of $\sigma_i(k)$, and the compared consensus errors, respectively.

From the simulation results employing the results of Corollary 1, we can see that outliers can affect the consensus performances of the second fourth agents and their neighboring agents. The variable $\sigma_2(k)$ can change dynamically when outliers of $k = 60$ occur such that the effect of outliers is weakened. For the fourth agent, the variable $\sigma_4(k)$ can change dynamically once outliers occur. Based on this merit, when outliers occur, the proposed control protocol in Equation (3) can have satisfactory performance. For the first, third, and fifth agents without the effect of outliers, variables $\sigma_l(k)$ $(l = 1, 3, 5)$ converge to zero. For the consensus error $x_{i,j}(k) - \frac{1}{N}\sum_{i=1}^{N} x_{i,j}(k)$ $(i = 1, 2, \ldots, 5; j = 1, 2)$, at $k = 60$ and $k = 80$, in terms of results in Corollary 1, consensus errors of the second and fourth agents and their neighboring agents have relatively large fluctuations. In light of the results in Theorem 2, the fluctuations of consensus errors of the second and fourth agents and their neighboring agents are very small. The simulation results demonstrate the effectiveness of the proposed control protocol Equation (3).

Moreover, to demonstrate the usefulness of the proposed control protocol in Equation (3), we choose the outliers obeying the zero mean Gaussian noise with covariance 10. The simulation results are shown in Figures 6–10. From the simulation results, we can see that the proposed control protocol has a satisfactory performance against the different types of measurement outliers.

CONCLUSION

In this paper, the observer-based H_∞ consensus problem for a class of linear multi-agent systems subject to measurement outliers is investigated. To mitigate the effect of measurement outliers, a novel control protocol based on an observer including a saturation function with variable saturation limits is proposed. Under the proposed observer-based control protocol, multi-agent systems can achieve the H_∞ consensus performance index over a finite horizon. Finally, by making a comparison with the general observer-based control protocol, the effectiveness of the proposed control protocol was verified in a simulation environment. Future research

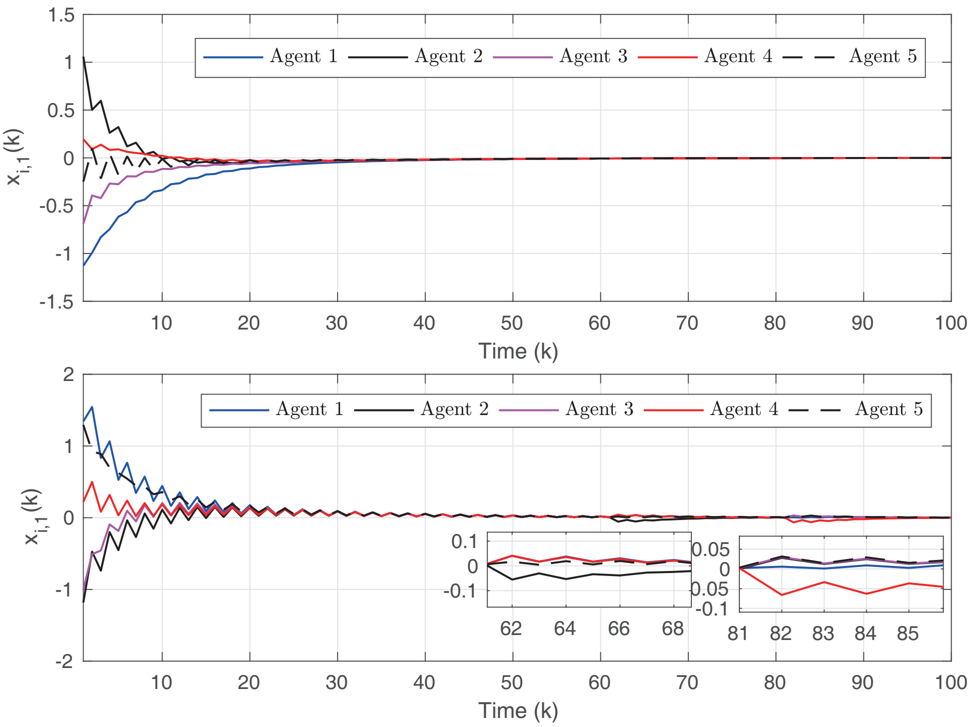

Figure 1. State trajectories $x_{i,1}(k)$ of five agents subject to measurement outliers (unit pulse).

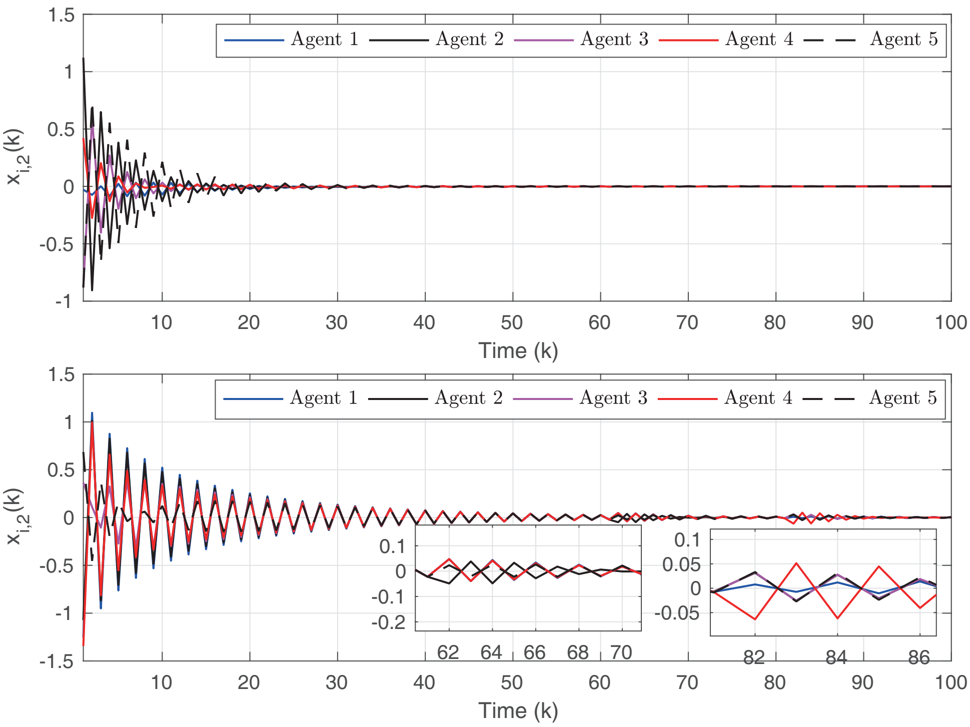

Figure 2. State trajectories $x_{i,2}(k)$ of five agents subject to measurement outliers (unit pulse).

directions may be devoted to an extension of the results obtained to nonlinear multi-agent systems and multi-agent systems with variable number of agents. Moreover, the obtained results can also be extended further to more practical research fields, such as sensor networks, neural networks, and multiple mobile robot system.

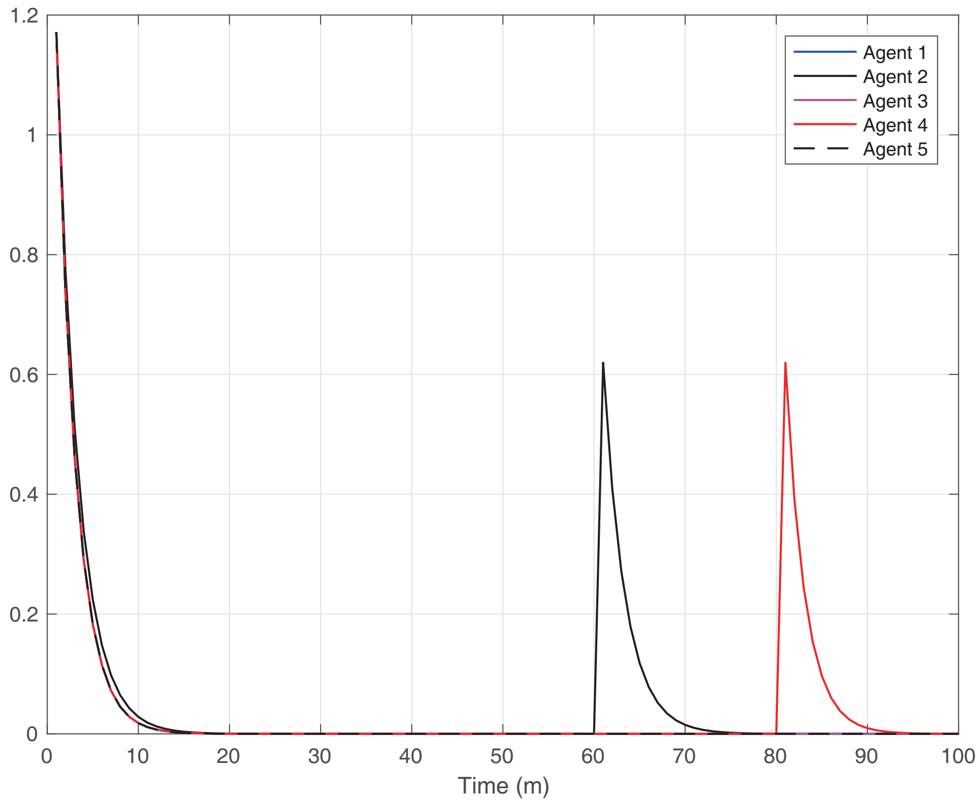

Figure 3. The profiles of $\sigma_i(k)$ of five agents subject to measurement outliers (unit pulse).

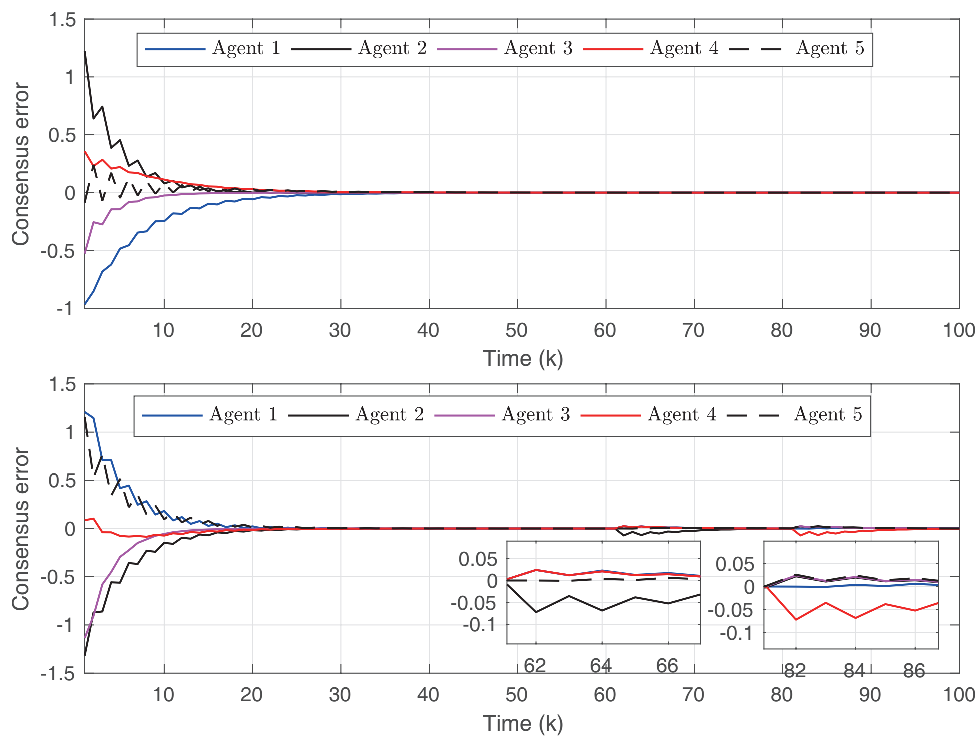

Figure 4. Consensus errors of $x_{i,1}(k) - \frac{1}{N}\sum_{i=1}^{N} x_{i,1}(k)$ subject to measurement outliers (unit pulse).

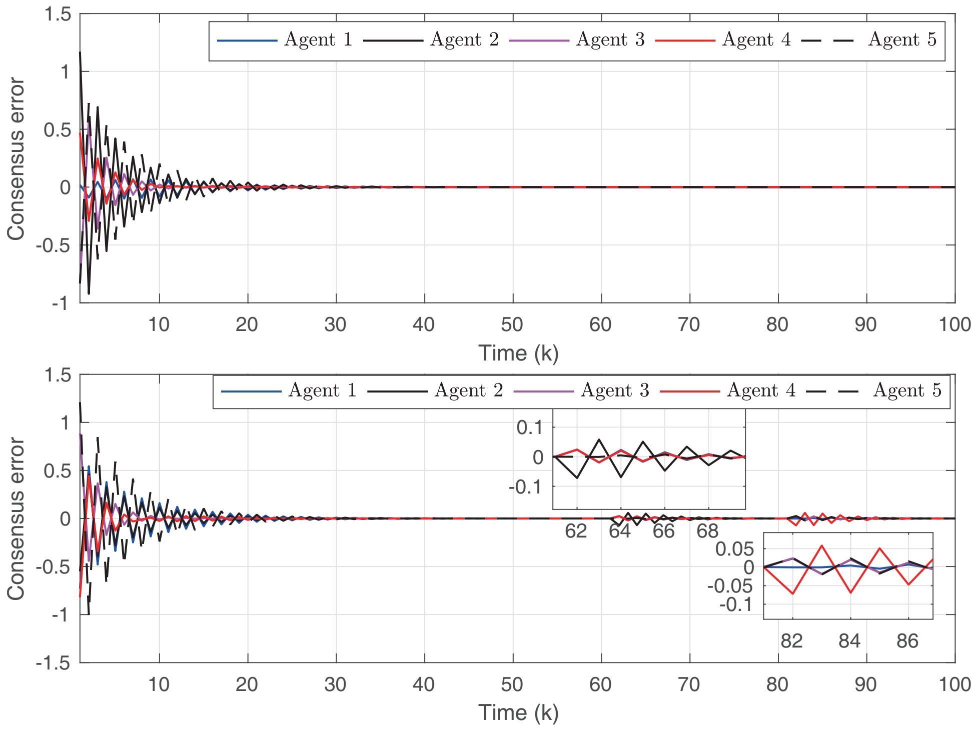

Figure 5. Consensus errors of $x_{i,2}(k) - \frac{1}{N}\sum_{i=1}^{N} x_{i,2}(k)$ subject to measurement outliers (unit pulse).

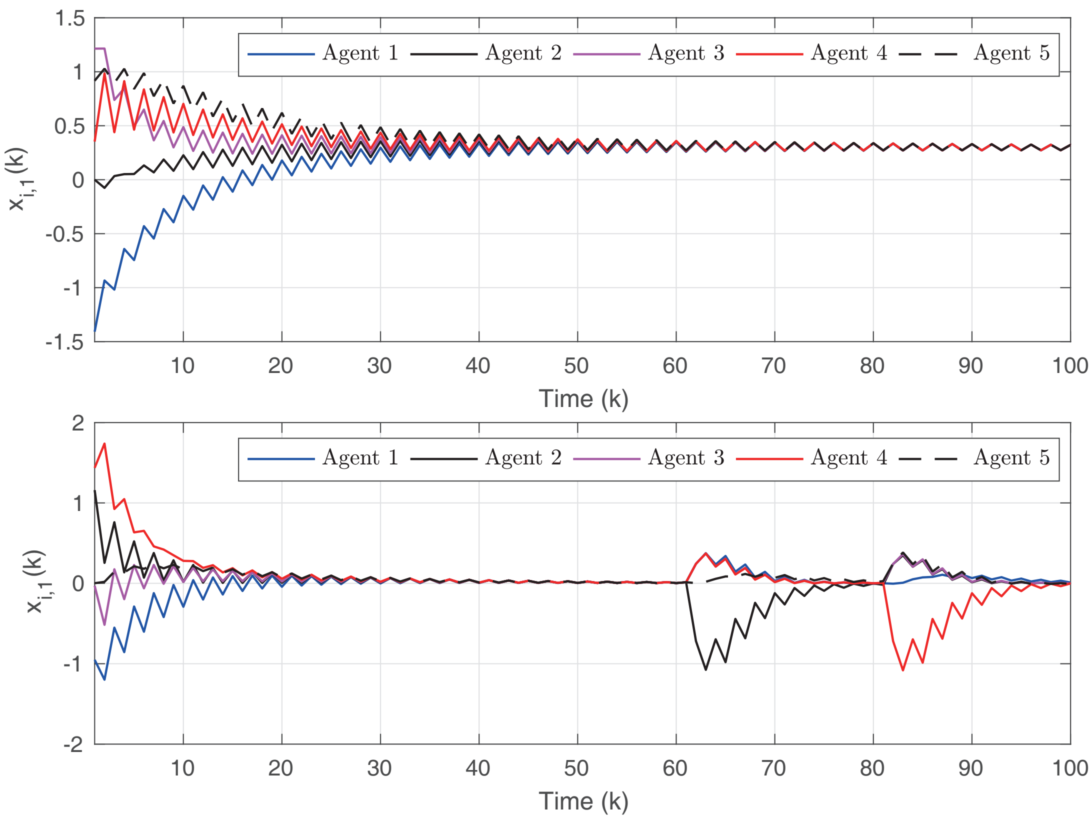

Figure 6. State trajectories $x_{i,1}(k)$ of five agents subject to measurement outliers (Gaussian noise).

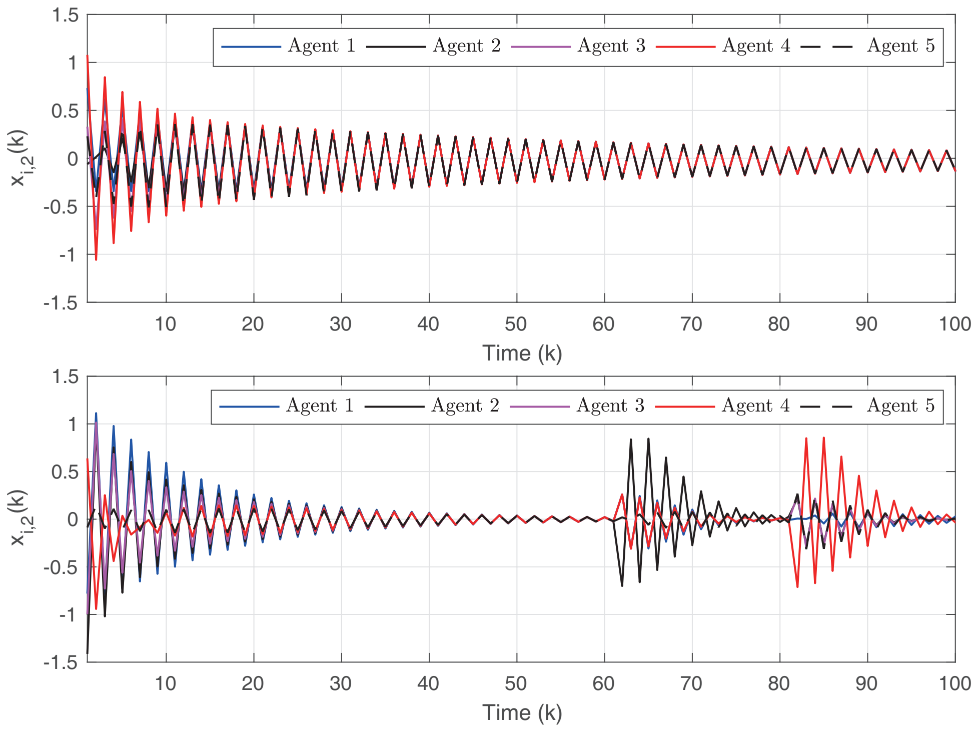

Figure 7. State trajectories $x_{i,2}(k)$ of five agents subject to measurement outliers (Gaussian noise).

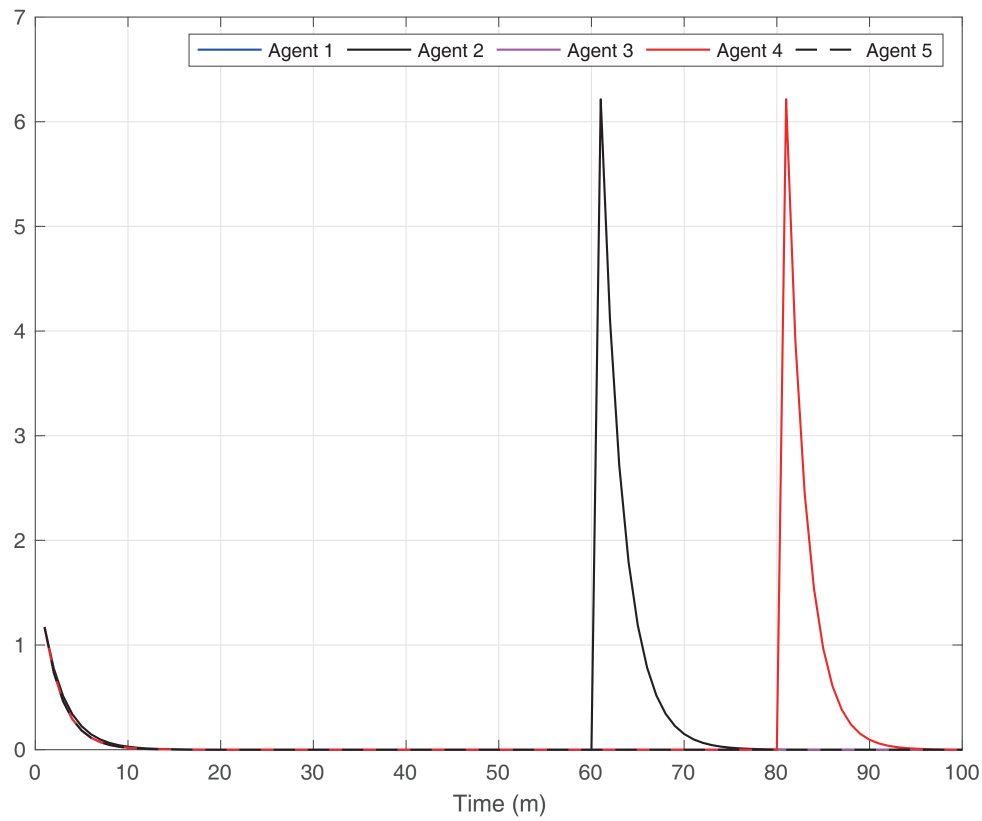

Figure 8. The profiles of $\sigma_i(k)$ of five agents subject to measurement outliers (Gaussian noise).

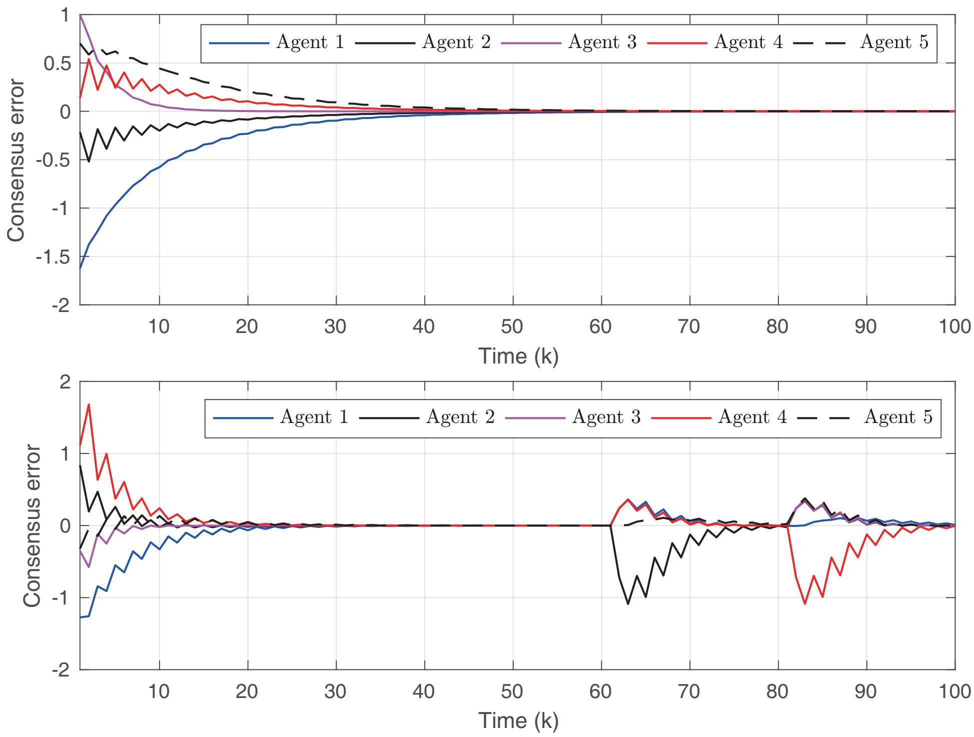

Figure 9. Consensus errors of $x_{i,1}(k) - \frac{1}{N}\sum_{i=1}^{N} x_{i,1}(k)$ subject to measurement outliers (Gaussian noise).

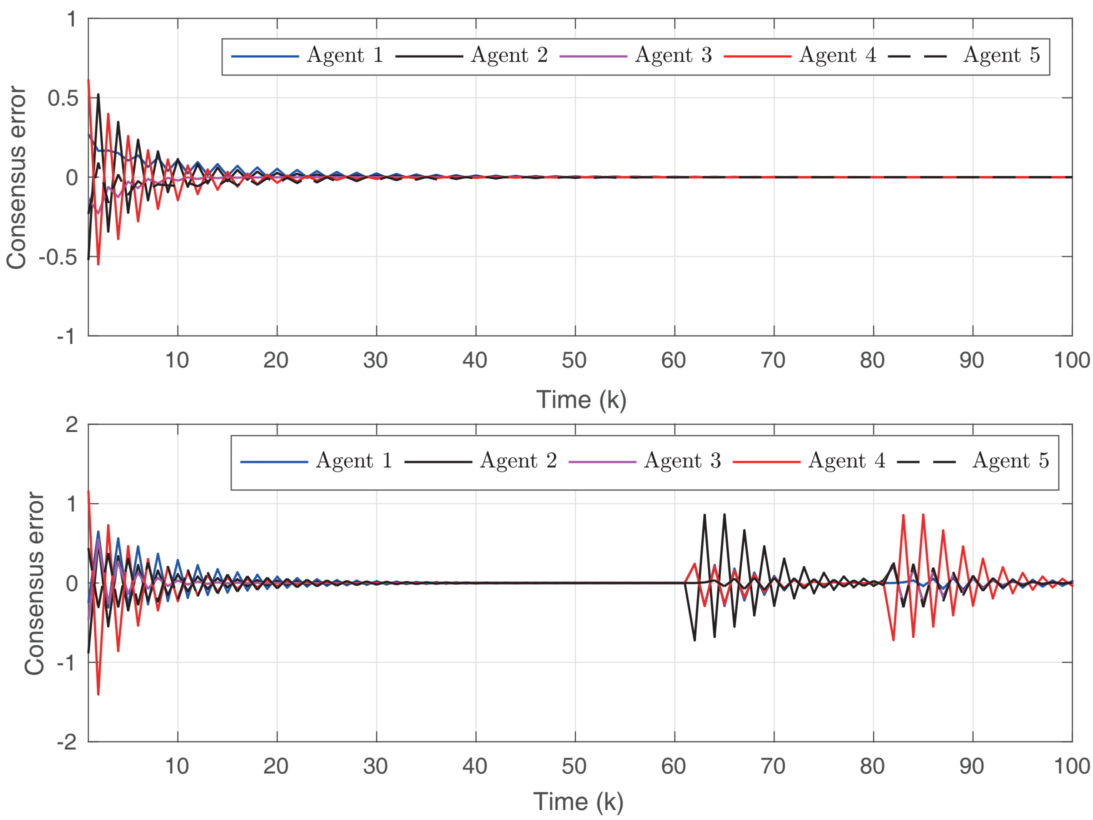

Figure 10. Consensus errors of $x_{i,2}(k) - \frac{1}{N}\sum_{i=1}^{N} x_{i,2}(k)$ subject to measurement outliers (Gaussian noise).

DECLARATIONS

Authors' contributions

Made substantial contributions to conception and design of the study and performed data analysis and interpretation: Lu Y, Zhang N

Performed data acquisition, as well as provided administrative, technical, and material support: Karimi HR

Availability of data and materials

Not applicable.

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Conflicts of interest

All authors declared that there are no conflicts of interest.

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Ethical approval and consent to participate

Not applicable.

Consent for publication

Not applicable.

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