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# Sampled-based bipartite tracking consensus of non-linear multiagents subject to input saturation

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## Abstract

This paper is concerned with the sampled-data bipartite tracking consensus problem for a class of nonlinear multi-agent systems (MASs) with input saturation. Both competitive and cooperative interactions coexist among agents in the concerned network. By resorting to Lyapunov stability theory and the linear matrix inequality (LMI) technique, several criteria are obtained to ensure that the considered MASs can achieve the bipartite tracking consensus. Besides, with the help of the decoupled method, the dimensions of LMIs are reduced for mitigation of the computation complexity so that the obtained results can be applied to large-scaled MASs. Furthermore, the controller gain matrix is explicitly expressed in terms of solutions to a set of LMIs. We also provide an estimate of elliptical attraction domain of bipartite tracking consensus. Finally, numerical simulation is exploited to support our theoretical analysis.

**Keywords:** Bipartite tracking consensus, sampled-data control, nonlinear multiagent systems, input saturation

## 1. INTRODUCTION

Over the last decade, adequate attention has been gained to various research on multiagent systems (MASs) due to pervasive applications in a variety of areas including, but are not limited to, coordination control of unmanned aerial vehicles<sup>[1]</sup>, formation control of multiple robot systems<sup>[2]</sup>, and dynamics of opinion forming<sup>[3]</sup>. Generally speaking, a MAS is composed of a group of agents, and these agents share local information



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from their neighbors through communication channels. The main objective of consensus problem for MASs is to construct an appropriate protocol so that agents can approach an agreement. Such a problem is one of the hottest topics in analysis and synthesis of cooperative behaviors. Thus far, most existing literature on MASs is concerned with consensus problem (see<sup>[4-15]</sup>). For example, the tracking consensus issue of single-integrator MASs was discussed in<sup>[10]</sup>, where a network-based consensus control protocol was designed to ensure that the followers' states reach an agreement on the leader's state. In<sup>[12]</sup>, the authors designed the consensus protocol for heterogeneous second-order nonlinear MASs with uniformly connected topologies in the presence of both uncertainties and disturbances.

It should be pointed out that communication linkages in the above-mentioned MASs are mainly concentrated on a cooperative network. However, in the real world, only cooperative linkages are insufficient to describe the intricate interactions among individuals. For example, in social networks, there are both competitive and cooperative relationships during the process of communication. In this case, the bipartite consensus entered the researchers' field of vision. The so-called bipartite consensus for MASs means that all agents reach a final state with an identical magnitude but opposite sign<sup>[16-20]</sup>. Recently, some scholars commit themselves to studying bipartite consensus problem of MASs with signed graphs, and there are some results on this topic scattered in the literature. For instance, based on state feedback and output feedback, the authors of<sup>[21]</sup> designed two distributed protocols to solve the bipartite output consensus problem for heterogeneous MASs with structurally balanced graphs. In<sup>[22]</sup>, the bipartite tracking consensus problem for linear MASs was investigated, in which the dynamic leader's control input was nonzero and unknown.

Notice that continuous information transmission among agents may cause a heavy burden and congestion in communication networks. Thus, it would lead to some difficulties in practical applications for some continuous-type consensus protocols. For this reason, the sampled-data control approach has been used to design various consensus protocols for MASs. The sampled-data control approach captures inherent properties of digital control, where the control input signal can be kept constant via a zero-order holder until the next sampling instant. Recently, many results have been reported concerning the sampled-data consensus of MASs<sup>[23-25]</sup>. Particularly, in<sup>[26]</sup>, with the aid of the input delay technique and decoupled method, the sampled-data consensus problem was investigated for nonlinear MASs with randomly occurring deception attacks. For a class of second-order MASs, an improved aperiodic sampled-data consensus protocol was designed in<sup>[27]</sup>, where only the sampled position data were exchanged among neighboring agents. Nevertheless, under competitive and cooperative communications, sampled-based bipartite tracking consensus problems are far from being adequately investigated due mainly to some difficulties aroused by a signed communication topology, which is one of the main motivations of this paper.

Actuator saturation is ubiquitous and unavoidable in practical engineering owing to various physical restrictions, such as power amplifiers and proportional valves<sup>[28]</sup>. The saturation phenomenon usually causes performance degradation, undesirable oscillatory behavior, and even instability<sup>[29,30]</sup>. Hence, it is critically important to take into account input constraints when designing distributed consensus protocols for MASs. Recent years have witnessed much research on consensus problems for MASs subject to input saturation<sup>[31-36]</sup>. For example, the containment control issue for MASs with bounded actuation was investigated in<sup>[33]</sup>, where an anti-windup compensation was designed by using convex conditions to improve the performance in the presence of actuator saturation. By considering the one-sided Lipschitz condition and input saturation, a new region of stability was provided in<sup>[37]</sup> to ensure the consensus error of one-sided Lipschitz nonlinear MASs was asymptotically stable. Based on the low-gain feedback technique, the authors of<sup>[38]</sup> studied the semi-global bipartite consensus problem for MASs with input saturation. However, a thorough literature search found that the research on the bipartite consensus problem for MASs is still in infancy, especially for the case when the MASs are involved in both signal-sampling and input saturation.

Inspired by the above discussions, we aim to further investigate the sampled-based bipartite tracking consensus of nonlinear MASs with input saturation. By resorting to Lyapunov stability theory and LMI technique, some criteria are established to ensure that the considered MASs can achieve the bipartite tracking consensus. The main contributions of this paper can be highlighted as follows.

- 1) The bipartite tracking consensus problem is investigated for a class of nonlinear MASs with signed communication topology. Both sampled-data control strategy and input saturation are taken into account in the design of bipartite tracking consensus protocol.
- 2) Some easy-to-check conditions are derived such that MASs under consideration can achieve the bipartite tracking consensus. On this basis, we give an estimate of upper bound of the sampling period. Meanwhile, the suitable controller is designed via solving a set of LMIs.
- 3) To reduce the computation burden, the matrix decoupling method is applied to reduce the dimensions of LMIs to be solved so that a lower computational effort is required for large-scaled MASs. In addition, an optimization method is also proposed to maximize the estimate of elliptical attraction domain of bipartite tracking consensus.

**Notation:** In this paper,  $\mathbb{R}^n$  and  $\mathbb{R}^{n \times m}$  denote, respectively, the  $n$ -dimensional Euclidean space and the set of all  $n \times m$  real matrices.  $X^T$  and  $X^{-1}$  represent, respectively, the transpose and inverse matrix of  $X$ . Let  $\underline{\lambda}(Y)$  and  $\bar{\lambda}(Y)$  be the smallest and the largest eigenvalue of the symmetric matrix  $Y$ , respectively. A symmetric matrix  $Y$  is positive-definite (negative-definite) if  $\underline{\lambda}(Y) > 0$  ( $\bar{\lambda}(Y) < 0$ ). Notation  $P > Q$  ( $P < Q$ ) indicates that  $P - Q$  is positive-definite (negative-definite). A nonnegative matrix  $P$  means that all the elements of  $P$  are nonnegative. For a matrix  $K$ ,  $K_{(i)}$  represents the  $i$ th row of  $K$ .  $\mathbb{I}_{[a,b]} = \{a, a + 1, a + 2, \dots, b\}$ , where  $a$  and  $b$  are nonnegative integers and  $b > a$ . For any  $\xi, \zeta \in \mathbb{R}^n$ ,  $\xi \leq \zeta$  implies  $\xi_{(i)} \leq \zeta_{(i)}$  ( $i \in \mathbb{I}_{[1,n]}$ ). Symbol  $*$  denotes a symmetric block in matrix expressions.

## 2. PROBLEM FORMULATION

A directed signed graph is denoted by  $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \mathcal{A}\}$ , where  $\mathcal{V} = \{1, 2, \dots, N\}$  represents the node set,  $\mathcal{E} = \{(p, q) | p, q \in \mathcal{V}, p \neq q\}$  denotes the edge set, and  $\mathcal{A} = (a_{pq})_{N \times N}$  is a weighted adjacent matrix. Edge  $(p, q) \in \mathcal{E}$  means that node  $p$  can communicate with node  $q$  directly. For the matrix  $\mathcal{A} = (a_{pq})_{N \times N}$ ,  $a_{pq} \neq 0$  means that  $(p, q) \in \mathcal{E}$ , and  $a_{pq} = 0$  otherwise. Especially,  $a_{pq} > 0$  implies the agents  $p$  and  $q$  have a cooperative relationship, and  $a_{pq} < 0$  indicates the antagonistic one. A directed path in graph  $\mathcal{G}$  from node  $p_0$  to node  $p_l$  means that there exists an ordered sequence of edges  $(p_0, p_1), (p_1, p_2), \dots, (p_{l-1}, p_l)$  such that  $(p_{k-1}, p_k) \in \mathcal{E}, k = 1, 2, \dots, l$ . A directed graph is said to have a directed spanning tree if there exists at least one node (called root node) which has directed paths to all the other nodes. The Laplacian matrix for signed graph  $\mathcal{G}$  is defined as

$$W = (w_{pq})_{N \times N}, \text{ where } w_{pq} = \begin{cases} -a_{pq}, & p \neq q \\ \sum_{q=1, q \neq p}^N |a_{pq}|, & p = q. \end{cases}$$

**Definition 1** <sup>[16]</sup> A directed signed graph  $\mathcal{G}$  is called structurally balanced, if the whole nodes in  $\mathcal{V}$  can be divided into two subsets  $\mathcal{V}_1$  and  $\mathcal{V}_2$ , which satisfy: (1)  $\mathcal{V}_1 \cup \mathcal{V}_2 = \mathcal{V}$  and  $\mathcal{V}_1 \cap \mathcal{V}_2 = \emptyset$ ; and (2) if  $p, q \in \mathcal{V}_k$  ( $k \in \{1, 2\}$ ), then  $a_{pq} \geq 0$ , and if  $p \in \mathcal{V}_k, q \in \mathcal{V}_{3-k}$  ( $k \in \{1, 2\}$ ), then  $a_{pq} \leq 0$ .

Let  $C_p \in \mathbb{R}^{p \times p}$  be the set of diagonal matrices whose diagonal elements are either 1 or  $-1$ .

**Lemma 1** <sup>[16]</sup> A signed graph  $\mathcal{G}$  of order  $N$  is structurally balanced if and only if there exists  $C = \text{diag}\{c_1, c_2, \dots, c_N\} \in C_N$  such that  $CA$  is a nonnegative matrix. In addition, the nodes can be divided into two subsets  $\mathcal{V}_1$  and  $\mathcal{V}_2$ , where  $\mathcal{V}_1 = \{l | c_l = 1\}$  and  $\mathcal{V}_2 = \{l | c_l = -1\}$ .

**Lemma 2** <sup>[39]</sup> Given a matrix  $\mathcal{M} = (m_{ij})_{N \times N}$ , where, for each  $i$ ,  $m_{ii} \geq 0$ ,  $m_{ij} \leq 0, \forall j \neq i$ , and  $\sum_{j=1}^N m_{ij} = 0$ , then  $\mathcal{M}$  has at least one zero eigenvalue, and all nonzero eigenvalues are in the open right half plane. Furthermore,  $\mathcal{M}$  has exactly one zero eigenvalue if and only if the directed graph of  $\mathcal{M}$  has a directed spanning tree.

Consider a group of agents with  $N$  followers (labeled by  $1, 2, \dots, N$ ) and one leader (labeled by  $0$ ). The dynamics of the  $k$ th follower is described by

$$\dot{s}_k(t) = A s_k(t) + f(s_k(t)) + B u_k(t), \tag{1}$$

and the leader is modeled by

$$\dot{s}_0(t) = A s_0(t) + f(s_0(t)), \tag{2}$$

where  $s_k(t) \in \mathfrak{X}^n (k \in \mathbb{I}_{[1,N]})$  and  $s_0(t) \in \mathfrak{X}^n$ , respectively, represent the state vectors of the  $k$ th follower and the leader;  $f : \mathfrak{X}^n \rightarrow \mathfrak{X}^n$  is an odd nonlinear vector-valued function denoting the inherent nonlinear dynamics of each agent;  $A \in \mathfrak{X}^{n \times n}$  and  $B \in \mathfrak{X}^{n \times m}$  are constant matrices; and  $u_k(t) \in \mathfrak{X}^m (k \in \mathbb{I}_{[1,N]})$  stands for the control input of the  $k$ th follower.

**Assumption 1** Assume that the communication graph  $\mathcal{G}$  is structurally balanced and has a spanning tree with the leader being the root. Besides, the network topology among followers is undirected.

**Definition 2** <sup>[22]</sup> The MAS in Equations (1) and (2) is said to achieve the bipartite tracking consensus, if there exists an appropriate distributed control scheme such that

$$\begin{cases} \lim_{t \rightarrow +\infty} \|s_k(t) - s_0(t)\| = 0, & \forall k \in \mathcal{V}_1, \\ \lim_{t \rightarrow +\infty} \|s_k(t) + s_0(t)\| = 0, & \forall k \in \mathcal{V}_2, \end{cases} \tag{3}$$

for any initial conditions  $s_k(0) \in \bar{\mathbb{S}}$ , ( $k \in \mathbb{I}_{[0,N]}$ ) ( $\bar{\mathbb{S}}$  represents an attraction domain of bipartite tracking consensus).

In this paper, we put forward to the following sampled-based controller (control scheme) of the  $k$ th follower subject to input saturation:

$$u_k(t) = \psi \left( -K \left( \sum_{l=1}^N |a_{kl}| (s_k(t_r) - \text{sign}(a_{kl}) s_l(t_r)) + |a_{k0}| (s_k(t_r) - \text{sign}(a_{k0}) s_0(t_r)) \right) \right), t \in [t_r, t_{r+1}), \tag{4}$$

where  $\{t_r\}_{r=0}^{+\infty}$  denotes a set of the sampling instants satisfying  $0 = t_0 < t_1 < t_2 < \dots < t_r < \dots$  and  $\lim_{r \rightarrow \infty} t_r = +\infty$ . The matrix  $K$  is the controller gain to be designed later. A standard saturation function  $\psi(\cdot) : \mathfrak{X}^m \rightarrow \mathfrak{X}^m$  satisfies  $\psi(x(t)) = (\text{sat}(x_1(t)), \text{sat}(x_2(t)), \dots, \text{sat}(x_m(t)))^T$  with  $\text{sat}(x_l(t)) = \text{sign}(x_l(t)) \min\{|x_l(t)|, 1\}$ .

According to Assumption 1 and Lemma 1, there exists  $\tilde{C} = \text{diag}\{c_0, c_1, c_2, \dots, c_N\} \in C_{N+1}$  such that the matrix  $\tilde{C} \mathcal{A} \tilde{C}$  is nonnegative. Then, the bipartite tracking consensus in Definition 2 is equivalent to

$$\lim_{t \rightarrow +\infty} \|s_k(t) - c_k s_0(t)\| = 0, k \in \mathbb{I}_{[1,N]}. \tag{5}$$

Set  $\eta_k(t) = s_k(t) - c_k s_0(t)$  and

$$z_k(t) = - \sum_{l=1}^N |a_{kl}| (s_k(t) - \text{sign}(a_{kl}) s_l(t)) - |a_{k0}| (s_k(t) - \text{sign}(a_{k0}) s_0(t)) (k \in \mathbb{I}_{[1,N]}).$$

From Equations (1) and (2) and the distributed control protocol in Equation (4), one can get that

$$\dot{\eta}_k(t) = A \eta_k(t) + \bar{f}(\eta_k(t)) + B \psi(K z_k(t_r)), t \in [t_r, t_{r+1}), \tag{6}$$

where  $\bar{f}(\eta_k(t)) = f(s_k(t)) - c_k f(s_0(t))$ . By means of the Kronecker product, the tracking error system in Equation (6) can be further turned into the following compact form:

$$\dot{\eta}(t) = (I_N \otimes A)\eta(t) + F(\eta(t)) + (I_N \otimes B)\psi((I_N \otimes K)z(t_r)), \quad t \in [t_r, t_{r+1}), \tag{7}$$

where

$$\begin{aligned} \eta(t) &= (\eta_1^T(t), \eta_2^T(t), \dots, \eta_N^T(t))^T, \\ z(t) &= (z_1^T(t), z_2^T(t), \dots, z_N^T(t))^T, \\ F(t) &= (\bar{f}^T(\eta_1(t)), \bar{f}^T(\eta_2(t)), \dots, \bar{f}^T(\eta_N(t)))^T. \end{aligned}$$

Since the leader has no neighbors, the Laplacian matrix of the MAS in Equations (1) and (2) can be partitioned as

$$W = \begin{bmatrix} 0_{1 \times 1} & 0_{1 \times N} \\ W_1 & W_2 \end{bmatrix},$$

where  $W_1 \in \mathfrak{R}^N$  and  $W_2 \in \mathfrak{R}^{N \times N}$ . From Assumption 1 and Lemmas 1 and 2, we deduce that all off-diagonal elements of  $\bar{W} = \tilde{C}W\tilde{C}$  are non-positive and matrix  $\bar{W}_2 = CW_2C$  is positive-definite, where  $C = \text{diag}\{c_1, c_2, \dots, c_N\}$ .

In this paper, we aim to design the sampled-based distributed protocol for the MAS in Equations (1) and (2) subject to input saturation and derive some conditions to ensure that the bipartite tracking consensus can be achieved.

### 3. MAIN RESULTS

Our main aim in this section is to establish some criteria to ensure that the considered MASs in Equations (1) and (2) can achieve the bipartite tracking consensus. Before proceeding, we give the following assumption and lemmas.

**Assumption 2** *The odd nonlinear vector-valued function  $f(\cdot)$  satisfies*

$$(f(x) - f(y))^T(f(x) - f(y)) \leq (x - y)^T \Lambda(x - y)$$

for all  $x, y \in \mathfrak{R}^n$ , where  $\Lambda$  is a positive-definite matrix.

**Lemma 3** <sup>[40]</sup> *Let  $\phi(a) = \psi(a) - a$  be the dead-zone function. Define the following associated set*

$$\mathcal{A}_n(a^0) = \{a, b \in \mathfrak{R}^n, -a^0 \leq a - b \leq a^0\},$$

where  $a^0 \in \mathfrak{R}^n$  and all elements of  $a^0$  are positive. Then, if  $a, b \in \mathcal{A}_n(a^0)$ , for any diagonal matrix  $\Phi > 0$ , the dead-zone function  $\phi(a)$  satisfies

$$\phi^T(a)\Phi(\phi(a) + b) \leq 0.$$

**Lemma 4** <sup>[41]</sup> *Let  $P$  be a positive semi-definite matrix and  $\gamma$  a positive scalar. Assume that the integration of vector-valued function  $\alpha(\cdot) : [0, \gamma] \rightarrow \mathfrak{R}^n$  is well-defined, then the following inequality holds*

$$\left( \int_0^\gamma \alpha(\theta) d\theta \right)^T P \left( \int_0^\gamma \alpha(\theta) d\theta \right) \leq \gamma \int_0^\gamma \alpha^T(\theta) P \alpha(\theta) d\theta.$$

**Lemma 5** <sup>[42]</sup> *Let matrices  $\Upsilon$  and  $\Xi$  be, respectively, positive-definite and symmetric, then for arbitrary  $x \in \mathfrak{R}^n$ ,*

$$\underline{\lambda}(\Upsilon^{-1}\Xi)x^T\Upsilon x \leq x^T\Xi x \leq \bar{\lambda}(\Upsilon^{-1}\Xi)x^T\Upsilon x.$$

**Lemma 6** <sup>[26]</sup> Let  $\chi(\cdot) : [0, +\infty) \rightarrow [0, 1]$  be a scalar function. Given symmetric constant matrices  $X_1, X_2, X_3$  with appropriate dimensions, then

$$X_1 + \chi(t)X_2 + (1 - \chi(t))X_3 < 0, \text{ if and only if } \begin{cases} X_1 + X_2 < 0 \\ X_1 + X_3 < 0. \end{cases}$$

In what follows, we always suppose  $P$  is a positive-definite matrix and denote  $\mathcal{P}(P, 1) = \{\eta \in \mathbb{R}^{nN} : \eta^T (I_N \otimes P)\eta \leq 1\}$ . In addition, let  $\mathbb{S} \triangleq \{\mathcal{P}(P, 1)\} \subset \bar{\mathbb{S}}$ .

The following theorem gives a condition to ensure the bipartite tracking consensus can be achieved for the MAS in Equations (1) and (2) with distributed protocol Equation (4).

**Theorem 1** Suppose Assumptions 1 and 2 hold. Let gain matrix  $K$  be given and  $\alpha$  and  $\beta$  be given positive scalars with  $\alpha > \beta$ . If there exist positive-definite matrix  $P$ , diagonal matrix  $\Phi > 0$ , matrix  $G$ , and positive constant  $\hbar$  such that the following LMIs hold

$$\begin{bmatrix} I_N \otimes P & (\bar{W}_{2(i)} \otimes (G + K)_{(j)})^T \\ * & 1 \end{bmatrix} \geq 0, \quad i \in \mathbb{I}_{[1, N]}, \quad j \in \mathbb{I}_{[1, m]}, \quad (8)$$

$$\begin{bmatrix} \Upsilon & 0 & I_N \otimes PB \\ * & -\beta I_N \otimes P & -\bar{W}_2 \otimes G^T \Phi \\ * & * & -2I_N \otimes \Phi \end{bmatrix} < 0 \quad (9)$$

with  $\Upsilon = I_N \otimes (PA + A^T P + \hbar PP + \hbar^{-1} \Lambda + \alpha P) - \bar{W}_2 \otimes (PBK + K^T B^T P)$ , then, for all initial conditions taken from  $\mathcal{P}(P, 1)$ , the MAS in Equations (1) and (2) with the distributed consensus scheme in Equation (4) can achieve the bipartite tracking consensus. In this case, a upper bound of the sampling period  $h$  can be estimated as

$$h < \frac{\alpha - \beta}{3\varpi_1 + \varpi_2} \quad (10)$$

with  $\varpi_1 = \bar{\lambda}(\bar{W}_2 \bar{W}_2^T)$ ,  $\varpi_2 = \bar{\lambda}(P^{-1}(BKA)^T PBKA) + \bar{\lambda}((BK)^T PBK) \bar{\lambda}(P^{-1} \Lambda) + \varpi_1 \bar{\lambda}(P^{-1} K^T K) \bar{\lambda}((BKB)^T PBKB)$ .

**Proof 1** Let  $\hat{\eta}(t) = (C \otimes I_n)\eta(t)$ . Considering the dead-zone function

$$\phi((I_N \otimes K)z(t)) = \psi((I_N \otimes K)z(t)) - (I_N \otimes K)z(t),$$

Equation (7) can be rewritten as

$$\begin{aligned} \dot{\hat{\eta}}(t) &= (I_N \otimes A)\hat{\eta}(t) + \hat{F}(\hat{\eta}(t)) + (C \otimes B)\psi((I_N \otimes K)z(t_r)) \\ &= (I_N \otimes A)\hat{\eta}(t) + \hat{F}(\hat{\eta}(t)) + (C \otimes BK)z(t_r) + (C \otimes B)\phi((I_N \otimes K)z(t_r)), \quad t \in [t_r, t_{r+1}), \end{aligned} \quad (11)$$

where  $\hat{F}(\hat{\eta}(t)) = (C \otimes I_n)F(\eta(t))$ .

Construct the following Lyapunov function:

$$V(t) = \hat{\eta}^T(t)(I_N \otimes P)\hat{\eta}(t). \quad (12)$$

Denote  $\Delta z(t) = z(t) - z(t_r)$ , for  $t \in [t_r, t_{r+1})$ . Taking the derivative of  $V(t)$  along the system in Equation (11), we have

$$\begin{aligned} \dot{V}(t) &= 2\hat{\eta}^T(t)(I_N \otimes P) \left[ (I_N \otimes A)\hat{\eta}(t) + \hat{F}(\hat{\eta}(t)) \right. \\ &\quad \left. + (C \otimes B)\phi((I_N \otimes K)z(t_r)) \right. \\ &\quad \left. + (C \otimes BK)z(t) - (C \otimes BK)\Delta z(t) \right]. \end{aligned} \quad (13)$$

According to Assumption 2 and  $2x^T y \leq \hbar x^T x + \hbar^{-1} y^T y$ , one gets that

$$\begin{aligned}
 & 2\hat{\eta}^T(t)(I_N \otimes P)\hat{F}(\eta(t)) \\
 & \leq \hbar\hat{\eta}^T(t)(I_N \otimes PP)\hat{\eta}(t) + \hbar^{-1}\hat{F}^T(\eta(t))\hat{F}(\eta(t)) \\
 & = \hbar\hat{\eta}^T(t)(I_N \otimes PP)\hat{\eta}(t) + \hbar^{-1}F^T(\eta(t))F(\eta(t)) \\
 & \leq \hbar\hat{\eta}^T(t)(I_N \otimes PP)\hat{\eta}(t) + \hbar^{-1}\eta^T(t)(I_N \otimes \Lambda)\eta(t) \\
 & = \hbar\hat{\eta}^T(t)(I_N \otimes PP)\hat{\eta}(t) + \hbar^{-1}\hat{\eta}^T(t)(I_N \otimes \Lambda)\hat{\eta}(t).
 \end{aligned} \tag{14}$$

For  $|a_{kl}|(s_k(t) - \text{sign}(a_{kl})s_l(t))$ , the following derivation holds:

$$\begin{aligned}
 & |a_{kl}|(s_k(t) - \text{sign}(a_{kl})s_l(t)) \\
 & = |a_{kl}|(s_k(t) - c_k s_0(t) + c_k s_0(t) - \text{sign}(a_{kl})s_l(t)) \\
 & = |a_{kl}|\eta_k(t) + |a_{kl}|(c_k s_0(t) - \text{sign}(a_{kl})s_l(t)).
 \end{aligned} \tag{15}$$

Noting that  $\tilde{C}\mathcal{A}\tilde{C}$  is nonnegative, one has: if  $a_{kl} > 0$ , then  $c_k c_l = 1$ , which implies that

$$|a_{kl}|(c_k s_0(t) - \text{sign}(a_{kl})s_l(t)) = a_{kl}(c_l s_0(t) - s_l(t)). \tag{16}$$

One the other hand, if  $a_{kl} < 0$ , then  $c_k c_l = -1$ , namely,  $c_k = -c_l$ . Hence,

$$|a_{kl}|(c_k s_0(t) - \text{sign}(a_{kl})s_l(t)) = -a_{kl}(-c_l s_0(t) + s_l(t)) = a_{kl}(c_l s_0(t) - s_l(t)). \tag{17}$$

From Equations (15)–(17), it follows that

$$\begin{aligned}
 z_i(t) & = -\sum_{l=1}^N |a_{kl}|(s_k(t) - \text{sign}(a_{kl})s_l(t)) - |a_{k0}|(s_k(t) - \text{sign}(a_{kl})s_0(t)) \\
 & = -\sum_{l=0}^N |a_{kl}|(s_k(t) - c_k s_0(t) + c_k s_0(t) - \text{sign}(a_{kl})s_l(t)) \\
 & = -\sum_{l=0}^N |a_{kl}|\eta_k(t) + \sum_{l=0}^N a_{kl}\eta_l(t),
 \end{aligned} \tag{18}$$

which implies that

$$z(t) = -(W_2 \otimes I_n)\eta(t). \tag{19}$$

It can also be validated that

$$\begin{aligned}
 & 2\hat{\eta}^T(t)(I_N \otimes P)(C \otimes BK)z(t) \\
 & = -2\hat{\eta}^T(t)(C \otimes PBK)(W_2 \otimes I_n)\eta(t) \\
 & = -2\hat{\eta}^T(t)(\bar{W}_2 \otimes PBK)\hat{\eta}(t).
 \end{aligned} \tag{20}$$

Note that

$$\begin{aligned}
 \Delta z(t) & = z(t) - z(t_r) \\
 & = \int_{t_r}^t \dot{z}(\theta) d\theta
 \end{aligned}$$

$$\begin{aligned}
&= - \int_{t_r}^t (W_2 \otimes I_n) \dot{\eta}(\theta) d\theta \\
&= - (W_2 \otimes I_n) \int_{t_r}^t [(I_N \otimes A)\eta(\theta) + F(\eta(\theta)) \\
&\quad + (I_N \otimes B)\psi((I_N \otimes K)z(t_r))] d\theta,
\end{aligned} \tag{21}$$

we obtain

$$\begin{aligned}
&-2\hat{\eta}^T(t)(C \otimes PBK)\Delta z(t) \\
&= 2\hat{\eta}^T(t)(\bar{W}_2 \otimes PBK) \int_{t_r}^t [(I_N \otimes A)\hat{\eta}(\theta) \\
&\quad + \hat{F}(\eta(\theta)) + (C \otimes B)\psi((I_N \otimes K)z(t_r))] d\theta.
\end{aligned} \tag{22}$$

From Lemmas 4 and 5, it follows

$$\begin{aligned}
&2\hat{\eta}^T(t)(\bar{W}_2 \otimes PBK) \int_{t_r}^t (I_N \otimes A)\hat{\eta}(\theta) d\theta \\
&\leq (t - t_r)\hat{\eta}^T(t)(\bar{W}_2\bar{W}_2^T \otimes P)\hat{\eta}(t) \\
&\quad + \frac{1}{t - t_r} \left( \int_{t_r}^t \hat{\eta}(\theta) d\theta \right)^T (I_N \otimes (BKA)^T PBKA) \left( \int_{t_r}^t \hat{\eta}(\theta) d\theta \right) \\
&\leq (t - t_r)\bar{\lambda}(\bar{W}_2\bar{W}_2^T)V(t) \\
&\quad + \int_{t_r}^t \hat{\eta}^T(\theta)(I_N \otimes (BKA)^T PBKA)\hat{\eta}(\theta) d\theta \\
&\leq (t - t_r)(\bar{\lambda}(\bar{W}_2\bar{W}_2^T)V(t) + \bar{\lambda}(P^{-1}(BKA)^T PBKA) \max_{\theta \in [t_r, t]} V(\theta)),
\end{aligned} \tag{23}$$

$$\begin{aligned}
&2\hat{\eta}^T(t)(\bar{W}_2 \otimes PBK) \int_{t_r}^t \hat{F}(\eta(\theta)) d\theta \\
&\leq (t - t_r)\hat{\eta}^T(t)(\bar{W}_2\bar{W}_2^T \otimes P)\hat{\eta}(t) \\
&\quad + \frac{1}{t - t_r} \left( \int_{t_r}^t \hat{F}(\eta(\theta)) d\theta \right)^T (I_N \otimes (BK)^T PBK) \left( \int_{t_r}^t \hat{F}(\eta(\theta)) d\theta \right) \\
&\leq (t - t_r)\bar{\lambda}(\bar{W}_2\bar{W}_2^T)V(t) \\
&\quad + \int_{t_r}^t \hat{F}^T(\eta(\theta))(I_N \otimes (BK)^T PBK)\hat{F}(\eta(\theta)) d\theta \\
&\leq (t - t_r)\bar{\lambda}(\bar{W}_2\bar{W}_2^T)V(t) \\
&\quad + \bar{\lambda}((BK)^T PBK) \int_{t_r}^t \hat{F}^T(\eta(\theta))\hat{F}(\eta(\theta)) d\theta \\
&= (t - t_r)\bar{\lambda}(\bar{W}_2\bar{W}_2^T)V(t) \\
&\quad + \bar{\lambda}((BK)^T PBK) \int_{t_r}^t F^T(\eta(\theta))F(\eta(\theta)) d\theta \\
&\leq (t - t_r)\bar{\lambda}(\bar{W}_2\bar{W}_2^T)V(t) \\
&\quad + \bar{\lambda}((BK)^T PBK) \int_{t_r}^t \hat{\eta}^T(\theta)(I_N \otimes \Lambda)\hat{\eta}(\theta) d\theta \\
&\leq (t - t_r)(\bar{\lambda}(\bar{W}_2\bar{W}_2^T)V(t) \\
&\quad + \bar{\lambda}((BK)^T PBK)\bar{\lambda}(P^{-1}\Lambda) \max_{\theta \in [t_r, t]} V(\theta)),
\end{aligned} \tag{24}$$



and

$$\begin{aligned}
 & 2\hat{\eta}^T(t)(\bar{W}_2 \otimes PBK) \int_{t_r}^t (C \otimes B)\psi((I_N \otimes K)z(t_r))d\theta \\
 & \leq (t - t_r)\hat{\eta}^T(t)(\bar{W}_2\bar{W}_2^T \otimes P)\hat{\eta}(t) \\
 & \quad + \frac{1}{t - t_r} \left( \int_{t_r}^t \psi((I_N \otimes K)z(t_r))d\theta \right)^T \\
 & \quad \times (I_N \otimes (BKB)^T PBKB) \int_{t_r}^t \psi((I_N \otimes K)z(t_r))d\theta \\
 & \leq (t - t_r)(\bar{\lambda}(\bar{W}_2\bar{W}_2^T)V(t) \\
 & \quad + \bar{\lambda}(\bar{W}_2\bar{W}_2^T)\bar{\lambda}(P^{-1}K^TK)\bar{\lambda}((BKB)^T PBKB) \max_{\theta \in [t_r, t]} V(\theta)). \tag{25}
 \end{aligned}$$

Substituting Equations (14), (20), and (23)–(25) into Equation (13), one has

$$\begin{aligned}
 \dot{V}(t) & \leq \hat{\eta}^T(t)(I_N \otimes (PA + A^T P + \hbar PP + \hbar^{-1}\Lambda))\hat{\eta}(t) \\
 & \quad - \hat{\eta}^T(t)(\bar{W}_2 \otimes (PBK + K^T B^T P))\hat{\eta}(t) \\
 & \quad + 2\hat{\eta}^T(t)(I_N \otimes PB)\hat{\phi}((I_N \otimes K)z(t_r)) \\
 & \quad + (t - t_r)(\bar{\lambda}(\bar{W}_2\bar{W}_2^T)V(t) + \bar{\lambda}(P^{-1}(BKA)^T PBKA) \max_{\theta \in [t_r, t]} V(\theta)) \\
 & \quad + (t - t_r)(\bar{\lambda}(\bar{W}_2\bar{W}_2^T)V(t) + \bar{\lambda}((BK)^T PBK)\bar{\lambda}(P^{-1}\Lambda) \max_{\theta \in [t_r, t]} V(\theta)) \\
 & \quad + (t - t_r)\bar{\lambda}(\bar{W}_2\bar{W}_2^T)(V(t) + \bar{\lambda}(P^{-1}K^TK)\bar{\lambda}((BKB)^T PBKB) \max_{\theta \in [t_r, t]} V(\theta)). \tag{26}
 \end{aligned}$$

Without loss of generality, we assume  $\eta(t_r) \in \mathcal{P}(P, 1)$ , that is,  $\eta^T(t_r)(I_N \otimes P)\eta(t_r) \leq 1$ . By using Schur complement to Equation (8), we obtain

$$(\bar{W}_{2(i)} \otimes (G + K)_{(j)})^T (\bar{W}_{2(i)} \otimes (G + K)_{(j)}) \leq I_N \otimes P, \tag{27}$$

which implies that

$$\begin{aligned}
 & \hat{\eta}^T(t_r)(\bar{W}_{2(i)} \otimes (G + K)_{(j)})^T (\bar{W}_{2(i)} \otimes (G + K)_{(j)})\hat{\eta}(t_r) \\
 & = \hat{\eta}^T(t_r)(\bar{W}_{2(i)} \otimes (G + K)_{(j)})^T c_i c_i (\bar{W}_{2(i)} \otimes (G + K)_{(j)})\hat{\eta}(t_r) \\
 & \leq \hat{\eta}^T(t_r)(I_N \otimes P)\hat{\eta}(t_r) \\
 & \leq 1. \tag{28}
 \end{aligned}$$

Furthermore, it follows from Equation (28) that

$$-1 \leq c_i(\bar{W}_{2(i)} \otimes (G + K)_{(j)})\hat{\eta}(t_r) \leq 1. \tag{29}$$

By Kronecker product, we can rewrite Equation (29) as

$$-\mathbf{1}_{nN} \leq (-W_2 \otimes (G + K))\eta(t_r) \leq \mathbf{1}_{nN}, \tag{30}$$

that is,

$$-\mathbf{1}_{nN} \leq (I_N \otimes (G + K))z(t_r) \leq \mathbf{1}_{nN}. \tag{31}$$

where  $\mathbf{1}_{nN} = (1, 1, \dots, 1)^T \in \mathfrak{R}^{nN}$ .

Hence, according to Lemma 3, we have

$$\begin{aligned} & 2\hat{\phi}^T((I_N \otimes K)z(t_r))(I_N \otimes \Phi)(\hat{\phi}((I_N \otimes K)z(t_r)) + (\bar{W}_2 \otimes G)\hat{\eta}(t_r)) \\ & = 2\phi^T((I_N \otimes K)z(t_r))(I_N \otimes \Phi)(\phi((I_N \otimes K)z(t_r)) - (I_N \otimes G)z(t_r)) \\ & \leq 0, \end{aligned} \tag{32}$$

where  $\hat{\phi}(\cdot) = (C \otimes I_n)\phi(\cdot)$ .

Subsequently, the inequality in Equation (26) in combination with Equation (32) indicates that

$$\begin{aligned} \dot{V}(t) & \leq \hat{\eta}^T(t)(I_N \otimes (PA + A^T P + \hbar PP + \hbar^{-1}\Lambda))\hat{\eta}(t) \\ & \quad - \hat{\eta}^T(t)(\bar{W}_2 \otimes (PBK + K^T B^T P))\hat{\eta}(t) \\ & \quad + 2\hat{\eta}^T(t)(I_N \otimes PB)\hat{\phi}((I_N \otimes K)z(t_r)) \\ & \quad - 2\hat{\phi}^T((I_N \otimes K)z(t_r))(I_N \otimes \Phi)(\hat{\phi}((I_N \otimes K)z(t_r)) + (\bar{W}_2 \otimes G)\hat{\eta}(t_r)) \\ & \quad + (t - t_r)(\bar{\lambda}(\bar{W}_2 \bar{W}_2^T)V(t) + \bar{\lambda}(P^{-1}(BKA)^T PBKA) \max_{\theta \in [t_r, t]} V(\theta)) \\ & \quad + (t - t_r)(\bar{\lambda}(\bar{W}_2 \bar{W}_2^T)V(t) + \bar{\lambda}((BK)^T PBK)\bar{\lambda}(P^{-1}\Lambda) \max_{\theta \in [t_r, t]} V(\theta)) \\ & \quad + (t - t_r)\bar{\lambda}(\bar{W}_2 \bar{W}_2^T)(V(t) + \bar{\lambda}(P^{-1}K^T K)\bar{\lambda}((BKB)^T PBKB) \max_{\theta \in [t_r, t]} V(\theta)) \\ & = \delta^T(t)\Upsilon_0\delta(t) + 3\varpi_1(t - t_r)V(t) + \beta V(t_r) + \varpi_2(t - t_r) \max_{\theta \in [t_r, t]} V(\theta), \end{aligned} \tag{33}$$

where  $\delta(t) = (\hat{\eta}^T(t), \hat{\eta}^T(t_r), \hat{\phi}^T((I_N \otimes K)z(t_r)))^T$  and

$$\Upsilon_0 = \begin{bmatrix} I_N \otimes (PA + A^T P + \hbar PP + \hbar^{-1}\Lambda) - \bar{W}_2 \otimes (PBK + K^T B^T P) & 0 & I_N \otimes PB \\ * & -\beta I_N \otimes P & -\bar{W}_2 \otimes G^T \Phi \\ * & * & -2I_N \otimes \Phi \end{bmatrix}.$$

According to Equation (9), one has

$$\begin{aligned} \dot{V}(t) & \leq -\alpha V(t) + 3\varpi_1(t - t_r)V(t) \\ & \quad + \beta V(t_r) + \varpi_2(t - t_r) \max_{\theta \in [t_r, t]} V(\theta) \\ & \leq -\alpha V(t) + 3\varpi_1 h V(t) \\ & \quad + \beta V(t_r) + \varpi_2 h \max_{\theta \in [t_r, t]} V(\theta), \quad t \in [t_r, t_{r+1}]. \end{aligned} \tag{34}$$

Next, we need to show that

$$\max_{\theta \in [t_r, t_{r+1}]} V(\theta) = V(t_r). \tag{35}$$

In fact, suppose that Equation (35) is not true, then there exists  $t^* \in [t_r, t_r + h]$  such that

$$V(t^*) > V(t_r). \tag{36}$$

In light of Equations (10) and (34), one has that

$$\dot{V}(t_r) \leq [-\alpha + \beta + 3\varpi_1 h + \varpi_2 h]V(t_r) < 0, \tag{37}$$

which means that there exists  $0 < \varepsilon < h$  such that

$$\dot{V}(\theta) < 0, \theta \in [t_r, t_r + \varepsilon). \tag{38}$$

Then, we get

$$V(\theta) < V(t_r), \theta \in (t_r, t_r + \varepsilon). \tag{39}$$

Let  $t^{**} \in [t_r + \varepsilon, t^*)$  satisfy

$$t^{**} = \inf\{t, t > t_r | V(t) = V(t_r)\}, \tag{40}$$

which further implies

$$\dot{V}(t^{**}) > 0. \tag{41}$$

In view of Equation (34), one obtains that

$$\begin{aligned} \dot{V}(t^{**}) &\leq -\alpha V(t^{**}) + \beta V(t^{**}) \\ &\quad + 3\varpi_1 h V(t^{**}) + \varpi_2 h V(t^{**}) \\ &\leq 0, \end{aligned} \tag{42}$$

which contradicts with Equation (41). Therefore,

$$\max_{\theta \in [t_r, t_{r+1}]} V(\theta) = V(t_r). \tag{43}$$

Furthermore, the inequality in Equation (34) in combination with Equation (43) indicates that

$$\dot{V}(t) \leq -\mu_1 V(t) + \mu_2 V(t_r), t \in [t_r, t_{r+1}), \tag{44}$$

where  $\mu_1 = \alpha - 3\varpi_1 h$  and  $\mu_2 = \varpi_2 h + \beta$ .

From the inequality in Equation (44), one gets

$$\frac{d}{dt} (e^{\mu_1 t} V(t)) \leq \mu_2 e^{\mu_1 t} V(t_r). \tag{45}$$

Integrating both sides of the inequality in Equation (45) from  $t_r$  to  $t$ , we have

$$e^{\mu_1 t} V(t) - e^{\mu_1 t_r} V(t_r) \leq \frac{\mu_2}{\mu_1} V(t_r) (e^{\mu_1 t} - e^{\mu_1 t_r}). \tag{46}$$

After a simple calculation, it is easy to see that

$$V(t) \leq \left[ \frac{\mu_2}{\mu_1} + \left(1 - \frac{\mu_2}{\mu_1}\right) e^{-\mu_1(t-t_r)} \right] V(t_r), t \in [t_r, t_{r+1}). \tag{47}$$

Setting  $\pi = \frac{\mu_2}{\mu_1} + \left(1 - \frac{\mu_2}{\mu_1}\right) e^{-\mu_1 h}$ , from Equation (10), one gets  $\mu_1 > \mu_2 > 0$ , which implies  $\pi < 1$ .

Hence,

$$V(t) \leq V(t_r), t \in [t_r, t_{r+1}). \tag{48}$$

It is easy to see that  $\eta(t) \in \mathbb{S}$ ,  $t \in [t_r, t_{r+1})$ .

For any  $t > 0$ , there exists a non-negative integer  $l$  such that  $t = lh + t_0$ , where  $0 \leq t_0 < h$ . Accordingly, one obtains that

$$\begin{aligned} V(t) &\leq \left[ \frac{\mu_2}{\mu_1} + \left( 1 - \frac{\mu_2}{\mu_1} \right) e^{-\mu_1(t-t_l)} \right] V(t_l) \\ &\leq \pi \left[ \frac{\mu_2}{\mu_1} + \left( 1 - \frac{\mu_2}{\mu_1} \right) e^{-\mu_1 t_0} \right] V(t_{l-1}) \\ &\leq \pi^2 \left[ \frac{\mu_2}{\mu_1} + \left( 1 - \frac{\mu_2}{\mu_1} \right) e^{-\mu_1 t_0} \right] V(t_{l-2}) \\ &\leq \dots \\ &\leq \pi^l \left[ \frac{\mu_2}{\mu_1} + \left( 1 - \frac{\mu_2}{\mu_1} \right) e^{-\mu_1 t_0} \right] V(t_0). \end{aligned} \tag{49}$$

Since  $\pi < 1$  and  $\frac{\mu_2}{\mu_1} + \left( 1 - \frac{\mu_2}{\mu_1} \right) e^{-\mu_1 t_0} \leq 1$ , we have

$$V(t) \leq V(0) \leq 1, t \geq 0. \tag{50}$$

Consequently, in view of Equation (49), we obtain that  $V(t) \rightarrow 0$  as  $t \rightarrow +\infty$ , which means that the MAS in Equations (1) and (2) achieves the bipartite tracking consensus based on the proposed consensus protocol in Equation (4). This completes the proof.

**Remark 1** Theorem 1 establishes a sufficient condition to ensure the bipartite tracking consensus for the concerned network. The condition is rather general, but it might have heavy computation burden for large-scale MASs. To reduce such computation burden, based on Theorem 1 and by utilizing the matrix decomposition technique, we derive the following theorem, giving a low-dimensional condition for the bipartite tracking consensus.

**Theorem 2** Let gain matrix  $K$  and positive scalars  $\alpha$  and  $\beta$  with  $\alpha > \beta$  be given. Under Assumptions 1 and 2, if there exist positive-definite matrix  $P$ , diagonal matrix  $\Phi > 0$ , matrix  $G$ , and positive constant  $\hbar$  such that the following LMIs hold

$$\begin{bmatrix} I_N \otimes P & (\bar{W}_2(i) \otimes (G + K)_{(j)})^T \\ * & 1 \end{bmatrix} \geq 0, \quad i \in \mathbb{I}_{[1,N]}, \quad j \in \mathbb{I}_{[1,m]}, \tag{51}$$

$$\begin{bmatrix} \Pi_1 - \lambda_i \Pi_2 & 0 & PB \\ * & -\beta P & -\lambda_i G^T \Phi \\ * & * & -2\Phi \end{bmatrix} < 0, \quad i \in \{1, N\}, \tag{52}$$

where  $\Pi_1 = PA + A^T P + \hbar PP + \hbar^{-1} \Lambda + \alpha P$ ,  $\Pi_2 = PBK + K^T B^T P$ , and  $\lambda_1$  and  $\lambda_N$  are, respectively, the minimum and maximum eigenvalues of  $\bar{W}_2$ , then, for all initial conditions taken from  $\mathcal{P}(P, 1)$ , the MAS in Equations (1) and (2) with the distributed consensus scheme in Equation (4) can achieve the bipartite tracking consensus. In this case, a upper bound of the sampling period  $h$  can be estimated as

$$h < \frac{\alpha - \beta}{3\varpi_1 + \varpi_2}. \tag{53}$$

**Proof 2** Clearly, it suffices to prove that the LMIs in Equation (52) imply the LMIs in Equation (9).

Since matrix  $\bar{W}_2$  is positive-definite, we can arrange its eigenvalues as  $0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_N$ . Based on matrix decomposition theory, there is the orthogonal matrix  $Q$  satisfying  $\bar{W}_2 = QUQ^T$ , where  $U = \text{diag}\{\lambda_1, \lambda_2, \dots, \lambda_N\}$ . Let  $\xi(t) = (Q^T \otimes I_n) \hat{\eta}(t)$ ; then, the inequality in Equation (33) can be rewritten as

$$\dot{V}(t) \leq \zeta^T(t) \Pi \zeta(t) - \alpha V(t) + 3\varpi_1(t - t_r) V(t) + \beta V(t_r) + \varpi_2(t - t_r) \max_{\theta \in [t_r, t]} V(\theta), \tag{54}$$

where  $\tilde{\phi}((I_N \otimes K)z(t_r)) = (Q^T \otimes I_n)\hat{\phi}((I_N \otimes K)z(t_r))$ ,  $\zeta(t) = (\xi^T(t), \xi^T(t_r), \tilde{\phi}^T((I_N \otimes K)z(t_r)))^T$ , and

$$\Pi = \begin{bmatrix} \Xi & 0 & I_N \otimes PB \\ * & -\beta I_N \otimes P & -U \otimes G^T \Phi \\ * & * & -2I_N \otimes \Phi \end{bmatrix}$$

with  $\Xi = I_N \otimes (PA + A^T P + \hbar PP + \hbar^{-1} \Lambda + \alpha P) - U \otimes (PBK + K^T B^T P)$ .

Denoting

$$\begin{aligned} \Omega &= \begin{bmatrix} \Pi_1 & 0 & PB \\ * & -\beta P & 0 \\ * & * & -2\Phi \end{bmatrix}, \\ \Omega_i &= -\lambda_i \begin{bmatrix} \Pi_2 & 0 & 0 \\ * & 0 & G^T \Phi \\ * & * & 0 \end{bmatrix}, \quad i \in \mathbb{I}_{[1,N]}, \\ \mathcal{I}_1 &= \underbrace{(1, 0, 0, \dots, 0)}_N, \mathcal{I}_2 = \underbrace{(0, 1, 0, \dots, 0)}_N, \dots, \mathcal{I}_N = \underbrace{(0, 0, 0, \dots, 1)}_N, \end{aligned}$$

one has

$$\Pi = \bar{\Gamma}^T \text{diag}\{\Omega + \Omega_1, \Omega + \Omega_2, \dots, \Omega + \Omega_N\} \bar{\Gamma}, \tag{55}$$

where  $\bar{\Gamma} = [\Gamma_1^T \Gamma_2^T \dots \Gamma_N^T]^T$  with  $\Gamma_i = \text{diag}\{\mathcal{I}_i \otimes I_n, \mathcal{I}_i \otimes I_n, \mathcal{I}_i \otimes I_m\}$ ,  $i \in \mathbb{I}_{[1,N]}$ .

In light of  $0 < \lambda_1 \leq \lambda_i \leq \lambda_N (i \in \mathbb{I}_{[2,N-1]})$ , there is a scalar function  $\chi(\cdot) : [0, +\infty) \rightarrow [0, 1]$  such that  $\lambda_i = (1 - \chi(t))\lambda_1 + \chi(t)\lambda_N (i \in \mathbb{I}_{[1,N]})$ . According to Lemma 6, we deduce from Equation (52) that  $\Omega + \Omega_i < 0$ , which implies  $\Pi < 0$ , namely, the LMI in Equation (9) is true. The proof is finished.

**Remark 2** Similar to the approach discussed in<sup>[15]</sup> for computational complexity, the computational complexity of the LMIs in Equation (9) in Theorem 1 can be represented as  $O((2n + m)N\mathcal{N}^3)$ , and that of the LMIs in Equation (52) in Theorem 2 can be expressed as  $O(2(2n + m)\mathcal{N}^3)$ , where  $\mathcal{N}$  is the total number of scalar decision variables. Clearly, compared to Theorem 1, the result of Theorem 2 is easy to implement for its low computation complexity.

We establish above sufficient conditions to ensure the MAS in Equations (1) and (2) can achieve the bipartite tracking consensus. Next, we consider the design problem of controller.

**Theorem 3** Under Assumptions 1 and 2, for given positive scalars  $\alpha > \beta$ , if there are positive scalar  $\hbar$ , positive-definite matrix  $\tilde{P}$ , diagonal positive-definite matrix  $\tilde{\Phi}$ , and matrices  $\tilde{G}, Y$  such that the following LMIs hold

$$\begin{bmatrix} I_N \otimes \tilde{P} & (\bar{W}_{2(i)} \otimes (\tilde{G} + Y)_{(j)})^T \\ * & 1 \end{bmatrix} \geq 0, \quad i \in \mathbb{I}_{[1,N]}, \quad j \in \mathbb{I}_{[1,m]}, \tag{56}$$

$$\begin{bmatrix} \tilde{\Pi}_1 - \lambda_i \tilde{\Pi}_2 & 0 & B\tilde{\Phi} & \tilde{P} \\ * & -\beta\tilde{P} & -\lambda_i \tilde{G}^T & 0 \\ * & * & -2\tilde{\Phi} & 0 \\ * & * & * & -\hbar\Lambda^{-1} \end{bmatrix} < 0, \quad i \in \{1, N\} \tag{57}$$

with  $\tilde{\Pi}_1 = A\tilde{P} + \tilde{P}A^T + \hbar I + \alpha\tilde{P}$  and  $\tilde{\Pi}_2 = BY + Y^T B^T$ , then, for every initial conditions belonging to  $\mathcal{P}(\tilde{P}^{-1}, 1)$ , the nonlinear MAS in Equations (1) and (2) can achieve bipartite tracking consensus. In this case, the controller gain matrix  $K$  can be designed as

$$K = Y\tilde{P}^{-1}, \tag{58}$$

and a upper bound of the sampling period  $h$  can be estimated as

$$h < \frac{\alpha - \beta}{3\varpi_1 + \tilde{\omega}_2} \quad (59)$$

with  $\tilde{\omega}_2 = q_1 + q_2 + q_3$ , in which  $q_1 = \bar{\lambda}(\tilde{P}(BY\tilde{P}^{-1}A)^T\tilde{P}^{-1}BY\tilde{P}^{-1}A)$ ,  $q_2 = \bar{\lambda}((BY\tilde{P}^{-1})^T\tilde{P}^{-1}BY\tilde{P}^{-1})\bar{\lambda}(\tilde{P}\Lambda)$ ,  $q_3 = \varpi_1\bar{\lambda}(Y^TY\tilde{P}^{-1})\bar{\lambda}((BY\tilde{P}^{-1}B)^T\tilde{P}^{-1}BY\tilde{P}^{-1}B)$ .

**Proof 3** Pre- and post-multiplying Equation (51) by  $\text{diag}\{I_N \otimes P^{-1}, 1\}$ , respectively, one gets that

$$\begin{bmatrix} I_N \otimes P^{-1} & \tilde{W}_{2(i)}^T \otimes (P^{-1}G^T + P^{-1}K^T)_{(j)} \\ * & 1 \end{bmatrix} \geq 0, \quad i \in \mathbb{I}_{[1,N]}, \quad j \in \mathbb{I}_{[1,m]}, \quad (60)$$

Selecting  $\tilde{P} = P^{-1}$ , from  $\tilde{G} = GP^{-1}$  and  $Y = KP^{-1}$ , we obtain Equation (56). Similarly, pre- and post-multiplying Equation (52) by  $\text{diag}\{P^{-1}, P^{-1}, \Phi^{-1}\}$ , respectively, and letting  $\tilde{\Phi} = \Phi^{-1}$ , we derive that

$$\begin{bmatrix} \tilde{\Pi}_1 + h^{-1}\tilde{P}\Lambda\tilde{P} - \lambda_i\tilde{\Pi}_2 & 0 & B\tilde{\Phi} \\ * & -\beta\tilde{P} & -\lambda_i\tilde{G}^T \\ * & * & -2\tilde{\Phi} \end{bmatrix} < 0, \quad i = \{1, N\}. \quad (61)$$

Using the Schur complement, the inequalities in Equation (61) are equivalent to the inequalities in Equation (57). Consequently, by Theorem 2, the bipartite tracking consensus is reached for the MAS in Equations (1) and (2).

In what follows, a corollary is presented to maximize an estimate of elliptical attraction domain of bipartite tracking consensus.

**Corollary 1** For the ellipsoidal set  $\mathbb{S}$ , the maximization problem for an estimate of ellipsoidal attraction domain of bipartite tracking consensus can be converted to minimization for matrix  $P$ , namely, maximization for matrix  $\tilde{P} = P^{-1}$ . This issue can be solved by using the following optimization problem:

$$\begin{cases} \min_{h, \tilde{P}, \tilde{\Phi}, \tilde{G}, Y} & \rho > 0 \\ \text{subject to:} & \\ & \text{Equation(56) - Equation(57) and } \begin{bmatrix} -\rho I & I \\ I & -\tilde{P} \end{bmatrix} \leq 0. \end{cases} \quad (62)$$

**Proof 4** The proof can be obtained directly from Theorem 2 and Schur complement, and is therefore omitted here.

**Remark 3** Since the maximization problem of ellipsoidal set  $\mathbb{S}$  is converted to the optimization problem in Equation (62), we can obtain a maximal ellipsoidal attraction region  $\mathcal{P}(P, 1)$  by utilizing the YALMIP toolbox in MATLAB. In addition, one of our future studies is to establish the relationship between the sampling interval and the maximal attraction domain of bipartite tracking consensus.

#### 4. SIMULATION STUDY

A simulation example is provided in this section to confirm the theoretical results.

Consider the MASs consisting of six agents, and the corresponding parameters are listed as follows:

$$A = \begin{bmatrix} -0.25 & -0.25 \\ 0.25 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \quad f(x(t)) = \begin{bmatrix} 0.3\sin(x_1(t)) \\ 0.2\tanh(x_2(t)) \end{bmatrix}.$$

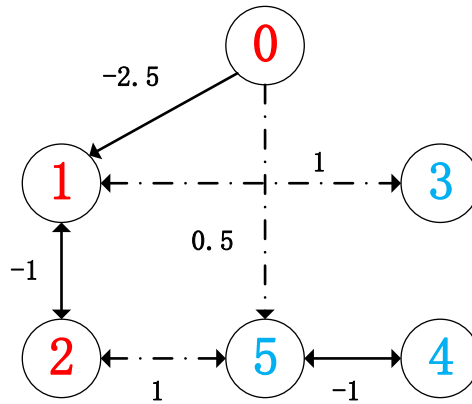


Figure 1. Communication topology.

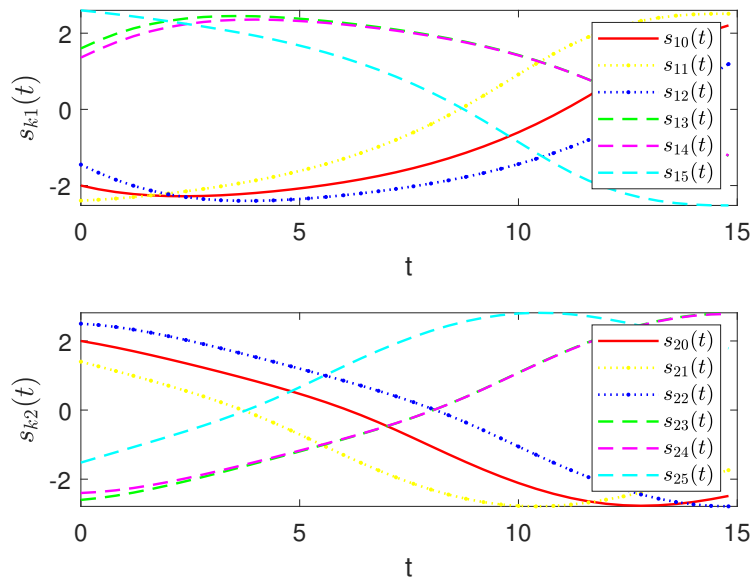


Figure 2. State evolutions of six agents without control.

Obviously, the nonlinear function  $f$  is odd and satisfies Assumption 2 with  $\Lambda = \text{diag}\{0.09, 0.04\}$ . The communication topology among the group of nonlinear agents is shown in Figure 1. Clearly, the corresponding Laplacian matrix is

$$W = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -2.5 & 4.5 & -1 & 1 & 0 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0.5 & 0 & 1 & 0 & -1 & 2.5 \end{bmatrix}.$$

It is easy to verify that the six agents can be divided into two clusters:  $\mathcal{V}_1 = \{0, 1, 2\}$  and  $\mathcal{V}_2 = \{3, 4, 5\}$ . Choose  $\alpha = 0.101$  and  $\beta = 0.001$ . By using YALMIP toolbox in MATLAB, we solve the optimal problem in Equation

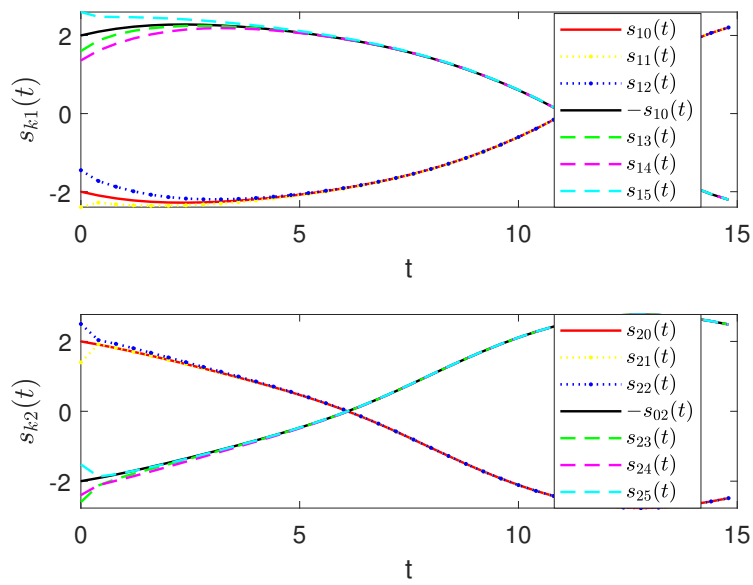


Figure 3. State evolutions of six agents with control.

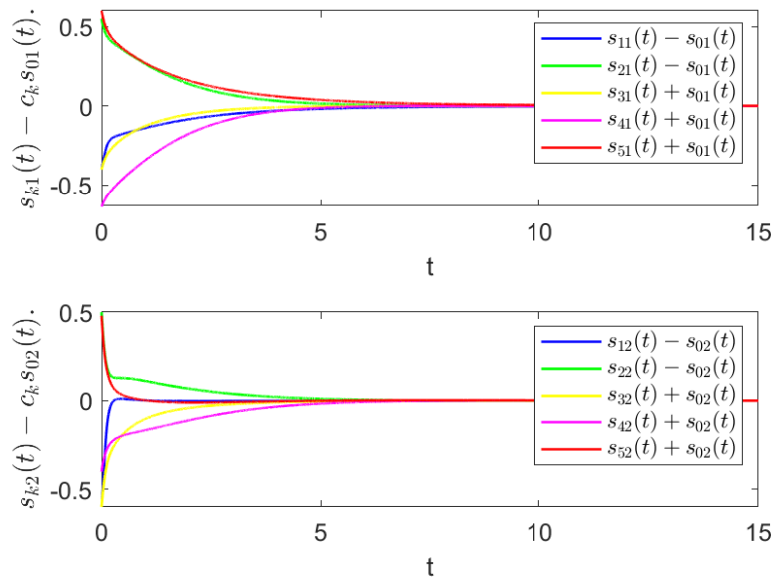


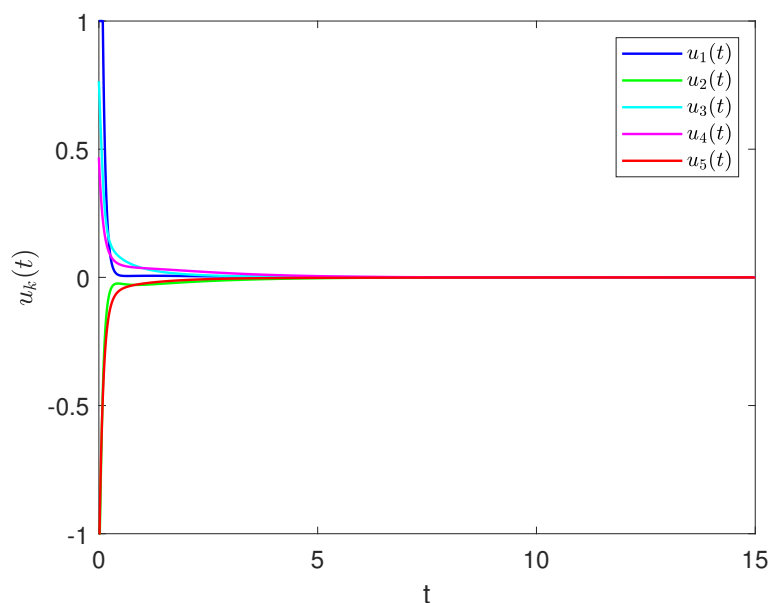
Figure 4. The bipartite tracking errors.

(62), and then obtain the feasible solution and controller gain matrix  $K$  as follows:

$$P = \tilde{P}^{-1} = \begin{bmatrix} 2.7078 & -1.2273 \\ -1.2273 & 3.1396 \end{bmatrix}, \quad K = Y\tilde{P}^{-1} = \begin{bmatrix} -0.0709 & 0.3658 \end{bmatrix}.$$

According to Theorem 3, the MAS in Equations (1) and (2) achieves the bipartite tracking consensus for any initial conditions  $s_k(0) \in \mathbb{S}$ . Furthermore, we obtain an upper bound of sampling period  $h < 5.6409 \times 10^{-2}$ .





**Figure 5.** The saturated control inputs.

We select sampling period  $h = 5 \times 10^{-2}$  and initial values  $s_0(0) = [-2, 2]^T$ ,  $s_1(0) = [-2.395, 1.4]^T$ ,  $s_2(0) = [-1.45, 2.5]^T$ ,  $s_3(0) = [1.6, -2.6]^T$ ,  $s_4(0) = [1.36, -2.4]^T$ , and  $s_5(0) = [2.6, -1.52]^T$ . The simulation results are shown in Figures 2–5. Figure 2 indicates the state evolution of each agent without control. It can be seen that the bipartite tracking consensus will not be achieved when there is no control for agents. Figure 3 plots the state evolution of each agent with sampled-based controller in Equation (4), while Figure 4 shows the bipartite tracking errors of followers. Figures 3 and 4 show that the MAS in Equations (1) and (2) with Equation (4) reaches bipartite tracking consensus, which is consistent with our theoretical result. Additionally, the saturated control inputs of followers are depicted in Figure 5.

## 5. CONCLUSIONS

In this paper, we have investigated the sampled-data tracking consensus problem for a class of nonlinear MASs subjected to input saturation over cooperation–competition networks. Based on the Lyapunov stability theory and some analysis tips, some LMI-based criteria are derived to guarantee the concerned MASs can reach the bipartite tracking consensus. Besides, by utilizing matrix decoupling method, the dimensions of LMIs are reduced to avoid a heavy computational burden. Moreover, an optimization problem is presented to maximize an estimate of ellipsoidal attraction domain of bipartite tracking consensus. Finally, a simulation example is provided to verify our main theoretical results. For the sampled-data-based bipartite tracking consensus of nonlinear MASs subject to input saturation, there are still some topics worthy of being investigated in the future, including the extension of our results to more general MASs with mixed time delays and other network-induced phenomena.

## DECLARATIONS

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### Authors' contributions

Conceptualization, Writing - original draft, Writing - review & editing: Yu L

Writing - original draft: Cui Y

Writing - review & editing: Lu Z

Methodology, Supervision: Liu Y

### Availability of data and materials

Not applicable.

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### Conflicts of interest

All authors declared that there are no conflicts of interest.

### Ethical approval and consent to participate

Not applicable.

### Consent for publication

Not applicable.

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