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# Synchronization for stochastic switched networks via delay feedback control

**Zebin Li**[1], **Fuying Cheng**[2], **Juan Tang**[3], **Hailing Dong**[1], **Jianliang Tang**[1]

[1]School of Mathematical Sciences, Shenzhen University, Shenzhen 518060, Guangdong, China.
[2]Shenzhen Pinghu No. 2 Experimental School, Shenzhen 518100, Guangdong, China.
[3]Pingshan School Affiliated to Sustech, Shenzhen 518118, Guangdong, China.

**Correspondence to:** Prof. Hailing Dong, Science of Mathematics College, Shenzhen University, 3688 Nanhai Avenue, Nanshan District, Shenzhen 518060, Guangdong, China. E-mail: hailing@szu.edu.cn

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## Abstract

In this paper, asymptotical synchronization in mean square, $H_\infty$-synchronization, and almost sure exponential synchronization are developed for a class of stochastic switched networks with Markov switching and Brown noise using a delay feedback controller that depends on the past state. By utilizing some inequality techniques, It$\hat{o}$ formula and Borel-Cantelli Lemma, we show that the stochastic switched network model can achieve asymptotical synchronization in mean square, $H_\infty$-synchronization, and almost sure exponential synchronization when the delay of the control is smaller than a given upper bound. Finally, the effectiveness of the theory is verified by a numerical simulation.

**Keywords:** Delay feedback control, Markov switching, asymptotical synchronization in mean square, almost sure exponential synchronization, $H_\infty$-synchronization

## 1. INTRODUCTION

Complex networks are ubiquitous in society and nature because they can abstractly describe almost all actual complex systems, such as the relationship between bacteria and cells, cooperation among academia, intelligent systems, Internet communication, biological engineering, power systems, Brownian motion, and others (see, e.g., [1–3] and reference therein). Therefore, they have become an irreplaceable framework and natural phenomenon in many complex social studies, which provides us with a new perspective and method of complexity research [4,5]. So, the research on complex networks is of great significance.

At present, the research content of complex networks mainly focuses on the following aspects: the formation mechanism of the network, the geometric nature of the network, the nature of the network model, the stability of the network, the synchronization and consistency of the network, and other issues [6–12]. Synchronization is a common collective behavior phenomenon in nature. In recent years, the synchronization theory and application research of complex networks have received extensive attention from domestic and foreign scholars. In terms of application research, synchronization mechanisms are applied to issues such as secure communications, nervous systems, superconducting materials, transportation networks, the Internet, and so on [13–17]. The most common synchronization problems in life are the glow of fireflies, the behavior of fish swimming in the ocean, and the gradual regular applause when the audience applauds. Therefore, the synchronization study of complex networks is of great significance in revealing the universal laws of network dynamics.

In the existing literature, people have studied various synchronization problems of the network model. In most of the literature, a commonly used and taken-for-granted method is to design a controller to synchronize the network. However, a more practical problem is the difference between the time to observe the state and the time to control the system. The time interval is greater than zero, so a more reasonable explanation is to set a controller with a time delay for the system. The paper [18] first proposed feedback control with time delays and applied it in the study of the stabilization of stochastic differential equations (SDEs). As we know, the delay feedback has not been used in the study of the synchronization issue for Markov switched stochastic networks. However, there are a large number of time-delay feedbacks in reality, such as monetary policy in economic systems, control systems in industrial processes, and neural systems in biology. Therefore, we study time-delay feedback control in Markov switching stochastic networks.

In addition, the existing research on Markov switching networks mainly focuses on the exponential mean square synchronization, such as [19]. Although exponential mean square synchronization provides convergence characteristics, it cannot guarantee that every orbit can be synchronized. In contrast, almost sure exponential synchronization has more advantages, because it not only ensures that all orbits are synchronized but also enables the synchronization speed to be faster in comparison. We not only study almost sure exponential synchronization but also investigate asymptotical synchronization in mean square and $H_\infty$-synchronization.

Based on the theory of SDEs, the properties of Markov processes, and the It$\hat{o}$ formula by designing a delay feedback controller, the almost sure exponent is established with stochastic synchronization of noisy complex networks. Next, we will introduce the contribution of this article:

1.We design a suitable controller with time delay to achieve almost sure exponential synchronization, asymptotical synchronization in mean square and $H_\infty$-synchronization in the Markov switched stochastic networks, which is more consistent with the actual situation and differs from the previous control strategies.

2.From a practical point of view, almost sure exponential convergence is almost certainly more effective, because it can ensure that each orbit of the stochastic process reaches convergence. The synchronization problem of complex networks with random noise and Markov switching under time-delay feedback control discussed in this paper is an infinite-dimensional problem, which is more difficult than a finite-dimensional problem.

3.Developing almost sure convergence in network synchronization is inherently challenging due to the need for estimating the time tail probability. As far as our current knowledge extends, there has not been any study addressing the almost sure convergence for complex networks involving a controller with time delay in the existing literature.

## 2.PROBLEM FORMULATION AND PRELIMINARIES

The notion diag $\{p_1, p_2, \cdots, p_n\}$ denotes the diagonal matrix with entries $p_1, p_2, \cdots, p_n$ on the diagonal. $I_N$ stands for the identity matrix with dimension $N$. $1_n$ is a $n$-dimension vector whose entries are 1. The notions $\lambda_{min}(\cdot), \lambda_{max}(\cdot)$ represent the minimum and maximum eigenvalue of a given matrix, respectively. For a vector $x$, let $x^\top$ denote the transpose vector and $\|x\|$ denote $L_2$-vector norm. Let $\mathbb{R}$ be the set of real numbers, $\mathbb{N} = \{1, 2, 3, \cdots\}$ be the set of natural numbers, $\mathbb{R}^n$ be the n-dimensional Euclidean space and $\mathbb{R}^{n\times n}$ be the set of all $n \times n$ real matrices. The symbol $\otimes$ denotes the standard Kronecker product.

Let $B_i(t) = [B_1(t), B_2(t), \ldots, B_m(t)]^T$ be an $\mathcal{F}_t$ -adapted Brownian motion and $\{\sigma(t), t \geq 0\}$ be a continuous time Markov process, where the state space of $\sigma(t)$ is $\mathbb{S} = \{1, 2, \cdots, m\}$. The generator $Q = (q_{ij})_{(m\times m)}$ of Markov process is given by

$$\mathrm{P}\{\sigma(t+\Delta t) = \ell \mid \sigma(t) = r\} = \begin{cases} q_{r\ell}\Delta t + o(\Delta t), & \text{if } \ell \neq r, \\ 1 + q_{r\ell}\Delta t + o(\Delta t), & \text{if } \ell = r, \end{cases}$$

where $\lim_{\Delta t\to 0^+} o(\Delta t)/\Delta t = 0$, $0 \leq q_{r\ell}, (\ell \neq r)$, and $q_{\ell\ell} = -\sum_{r=1,r\neq\ell}^{m} q_{r\ell}$.

Assume that $\sigma(\cdot)$ and $B(\cdot)$ are independent of each other. By using a delay pinning feedback controller, we study the Markovian switched stochastic network as follows:

$$dx_i(t) = h(x_i(t), \sigma_t)dt + r(x_i(t), \sigma_t)dB(t) + c\sum_{j=1}^{M} a_{ij}(\sigma_t)x_j(t)dt + u(x_i(t-\tau), \sigma_t)dt, \tag{1}$$

$$u(x_i(t-\tau), \sigma_t) = -\rho d_i(\sigma_t)(x_i(t-\tau) - s(t-\tau)), \tag{2}$$

where $i = 1, 2, 3, \cdots, M, x_i(t) \in \mathbb{R}^n$ represents the state vector of the $i$th node; $h(x_i(t), \sigma_t)$ and $r(x_i(t), \sigma_t, t)$ are continuous functions, and they separately describe the dynamics and noise intensity. The coupling matrix $A(\sigma_t) = (a_{ij}(\sigma_t))_{N\times N}$ is irreducible, which also satisfies: $a_{ij}(\sigma_t) \geq 0$, $a_{ij}(\sigma_t) = a_{ji}(\sigma_t)$, $i \neq j$ and $a_{ii}(\sigma_t) = -\sum_{j=1,j\neq i}^{M} a_{ij}(\sigma_t)$; where $d_i(\sigma_t) = I_{i\in\mathcal{D}(\sigma_t)}$ is the indication function for the pinned node subset $\mathcal{D}(\sigma_t) \subset \{1, 2, \cdots, N\}$, $\rho$ is the control gain, $u(0, i, t) \equiv 0$. The initial data are $\{x(t) : -\tau \leq t \leq 0\} = \xi \in C([-\tau, 0]; R^n)$ and $\sigma(0) = \sigma_0 \in S$.

Denote $s(t)$ as a desirable state solution that satisfies:

$$ds(t) = H(s(t), \sigma_t)dt + R(s(t), \sigma_t)dB(t), \tag{3}$$

The set $s(t) \times 1_n$ is used as the synchronization manifold. The initial value of $s(t)$ is given by $s(t) = \psi(t) \in C^b_{\mathcal{F}_0}([-\tau, 0], \mathbb{R}^n)$. The error system of this network can be written as follows:

$$de(t) = \hat{H}(e(t), \sigma_t)dt + \hat{R}(e(t), \sigma_t)dB(t) + cA(\sigma_t)e(t)dt - \rho D(\sigma_t)e(t-\tau)dt, \tag{4}$$

where: $e(t) = (e_1^\top(t), \cdots, e_N^\top(t))^\top$, $\hat{H}(e(t), \sigma_t) = H(x(t), \sigma_t) - H(s(t), \sigma_t)$, $\hat{R}(e(t), \sigma_t) = R(x(t), \sigma_t) - R(s(t), \sigma_t)$. $H(x(t), \sigma_t) = (h(x_1(t), \sigma_t), \cdots, h(x_N(t), \sigma_t))^\top$, $R(x(t), \sigma_t) = (r(x_1(t), \sigma_t), \cdots, r(x_N(t), \sigma_t))^\top$, $H(s(t), \sigma_t) = (h(s(t), \sigma_t) \times 1_N)^\top$, $R(s(t), \sigma_t) = (h(s(t), \sigma_t) \times 1_N)^\top$, $D(\sigma_t)$ =diag$[d(\sigma_t), d(\sigma_t), \cdots, d(\sigma_t)]$.

**Definition 1.** [20] ($H_\infty$-Synchronization) The network (4) achieves $H_\infty$-synchronization, if we have

$$\int_0^\infty E\|e(t)\|^2 dt < \infty.$$

**Definition 2.** [20] (Synchronization in Mean Square) The solution of the network (4) satisfies

$$\lim_{t\to+\infty} E\|e(t)\|^2 = 0,$$

for $\forall\,\psi(t)$, namely, the network (1) achieves synchronization in mean square.

**Definition 3.**[16] (Exponential Synchronization) System (1) is said to achieve exponential synchronization, if there exist constants $\varepsilon > 0$ and $G > 0$ such that

$$E\|e(t)\|^2 \leq Ge^{-\varepsilon t}.$$

**Definition 4.**[16] (Almost Surely Exponentially Synchronization) The solution of the network (1) with the initial has the property that:

$$\limsup_{t\to\infty} \frac{1}{t}\log(\|e(t)\|) < 0, \qquad a.s.$$

That is, the network (1) is almost surely exponentially synchronization.

**Definition 5.** ($QUAD$ Condition) The function $h(x,\ell)$ is said to satisfy the $QUAD$ condition, denoted as $h(x,\ell) \in QUAD(P_\ell, \Delta_\ell, \varphi_\ell)$, if we can find positive define diagonal matrices $P_\ell$ and $\Delta_\ell$, for $\ell = 1, 2, \cdots, m$ and a positive constant $\varphi_\ell$, such that for any $x, y \in \mathbb{R}^n$, the following condition holds:

$$(x-\bar{x})^\top P_\ell[h(x,\ell) - h(\bar{x},\ell) - \Delta_\ell(x-\bar{x})] \leq -\varphi_\ell(x-\bar{x})^\top P_\ell(x-\bar{x}).$$

**Assumption 1.**[21] If we can find $\eta_\ell > 0$ and $\alpha_\ell > 0$ such that

$$\|h(x,\ell) - h(\bar{x},\ell)\|^2 \leq \eta_\ell(\|x-\bar{x})\|^2),$$
$$\|r(x,\ell) - r(\bar{x},\ell)\|^2 \leq \alpha_\ell(\|x-\bar{x})\|^2).$$

**Assumption 2.** [9,18] The function $r(x,\ell)$ satisfies the Lipschitz condition and we can find $\omega_l > 0$ such that for $l = 1, 2, \ldots, N$,

$$\text{trace}[r(x,\ell) - r(\bar{x},\ell)]^T[r(x,\ell) - r(\bar{x},\ell)] \leq \omega_\ell(x-\bar{x})^2.$$

**Assumption 3.** [20] If we can find $\gamma > 0$ such that

$$\|u(x,\ell) - u(\bar{x},\ell)\| \leq \gamma\|x-\bar{x}\|,$$

for all $(x,\ell) \in R^n$ and $t \geq 0$. This assumption, together with $u(0,\ell) \equiv 0$, implies

$$\|u(x,\ell)\| \leq \gamma\|x\|.$$

**Assumption 4.** [22] If we can find $K > 0$ such that

$$\|h(x,\ell)\| \leq K(1+\|x\|) \; and \; \|r(x,\ell)\| \leq K(1+\|x\|), \tag{5}$$

for all $(x,\ell) \in R^n \times S$.

**Assumption 5.**[20] If we can choose functions $W$ and $\Lambda$, as well as a positive number $c$ and $q \geq 2$, such that

$$|x|^2 \leq W(x,\ell) \leq \Lambda(x) \quad \forall (x,\ell) \in R^n \times S, \tag{6}$$

and

$$\begin{aligned}\mathcal{L}W(x,\ell) :=& W_t(x,\ell) + W_x(x,\ell)u(x,\ell) + \frac{1}{2}trace[R^T(x,\ell)\mathbf{P}(l)r(x,\ell)] + \frac{1}{2}\sum_{l=1}^{m} q_{\sigma_t l}e^T(t)\mathbf{P}(l)e(t)\\ \leq& -c\Lambda(x),\end{aligned} \tag{7}$$

for all $(x,\ell) \in R^n \times S$, $W \in C^{2,1}(R^n \times S; R_+)$, $\Lambda \in C(R^n \times [-\tau,\infty))$.

## 3 SYNCHRONIZATION ANALYSIS

For studying the problem of the synchronization of the controlled complex network system (4), we define two segments: $\hat{e}_t := e(t+s) : -\tau \leq s \leq 0$ and $\hat{\sigma}_t := \sigma(t+s) : -\tau \leq s \leq 0$ for $t \geq 0$. For $\hat{e}_t$ and $\hat{\sigma}_t$ to be well defined for $0 \leq t \leq \tau$, we set $e(s) = e_0$ and $\sigma_s = \sigma_0$ for $s \in [-\tau, 0)$ .

We choose the Lyapunov-Krasovskii function as follows:

$$V(\hat{e}_t, \hat{\sigma}_t) = U(e(t), \sigma_t) + \frac{2\gamma^2}{\beta} \int_{-\tau}^{0} \int_{t+s}^{t} [\tau \|\hat{H}(e(v), \sigma_v) + cA(\sigma_v)e(v) - \rho D(\sigma_v)e(v-\tau)\|^2 + \|\hat{R}(e(v), \sigma_v)\|^2] dv ds, \tag{8}$$

where for $t \geq 0$, $U(e(t), \sigma_t) = \frac{1}{2} e^{\top}(t) \mathbf{P}(\sigma_t) e(t)$, $\mathbf{P}(\sigma_t) = I_N \otimes P(\sigma_t)$ .

We claim that $V(\hat{e}_t, \hat{\sigma}_t)$ is an Itô process on $t \geq 0$, In fact, according to the generalized Itô formula [22], we have

$$dV(e(t), \sigma_t) = \mathcal{L}V(e(t), \sigma_t)dt + dM(t). \tag{9}$$

For $t \geq 0$, where $M(t)$ is a martingale, with the initial value is 0, and

$$\mathcal{L}V(\hat{e}_t, \hat{\sigma}_t) = \mathcal{L}U(e(t), \sigma_t) + I(t). \tag{10}$$

where $I(t) = U_x(e(t), \sigma_t)[u(e(t-\tau), \sigma_t) - U(e(t), \sigma_t)] + \frac{2\gamma^2}{\beta}\tau[\tau\|\hat{H}(e(t), \sigma_t) + cA(\sigma_t)e(t) - \rho D(\sigma_t)e(t-\tau)\|^2 + \|\hat{R}(e(t), \sigma_t)\|^2] - \frac{2\gamma^2}{\beta}\int_{t-\tau}^{t}[\tau\|\hat{H}(e(v), \sigma_v) + cA(\sigma_v)e(v) - \rho D(\sigma_v)e(v-\tau)\|^2 + \|\hat{R}(e(v), \sigma_v)\|^2]dv$.

**Lemma 1.** Under Assumptions (1)-(6), then we have the following inequality:

$$\mathcal{L}U(e(t), \sigma_t) \leq \left( -\rho\bar{p}\underline{\lambda} - \lambda_1 + \frac{\bar{p}\omega}{2} + \pi \right) \|e(t)\|^2,$$

where $\lambda_1 = \lambda_{min}[\varphi \mathbf{P}(\sigma_t) - I_N \otimes P(\sigma_t)\Delta]$ .

Proof: Step 1, for $u(x, \ell)$, we will let $U_t(x, \ell) = \frac{\partial u(x,\ell)}{\partial t}$,$U_x(x, \ell) = \left( \frac{\partial u(x,\ell)}{\partial x_1}, \cdots, \frac{\partial u(x,\ell)}{\partial x_n} \right)$, and $\mathcal{L}U : R^n \times S \longrightarrow R$ is defined by

$$\begin{aligned}\mathcal{L}U(e(t), \sigma_t) =& U_x(e(t), \sigma_t)U(e(t), \sigma_t) + e^{\top}(t)\mathbf{P}(\sigma_t)[\hat{H}(e(t), \sigma_t) + cA(\sigma_t)e(t) - \rho D(\sigma_t)e(t-\tau)] \\ &+ \frac{1}{2} tr[\hat{R}^{\top}(e(t), \sigma_t)\mathbf{P}(\sigma_t)\hat{R}(e(t), \sigma_t)] + \frac{1}{2}\sum_{l=1}^{m} q_{\sigma_t l} e^{T}(t)\mathbf{P}(l)e(t) \\ \triangleq& \mathcal{L}U_1(t) + \mathcal{L}U_2(t) + \mathcal{L}U_3(t) + \mathcal{L}U_4(t).\end{aligned} \tag{11}$$

Step 2, we compute the $\mathcal{L}U_1(t)$–$\mathcal{L}U_4(t)$ respectively. For the $\mathcal{L}U_1(t)$, $U_x(e(t), \sigma_t) = e^{\top}(t)\mathbf{P}(\sigma_t)$, $U(e(t), \sigma_t) = -\rho D(\sigma_t)e(t)$, we obtain that

$$\mathcal{L}U_1(t) \leq -\rho\bar{p}\underline{\lambda}\|e(t)\|^2, \tag{12}$$

where $\bar{p} = \lambda_{max}\{P(\sigma_t)\}$, $\underline{p} = \lambda_{min}\{P(\sigma_t)\}$, $\bar{\lambda} = \lambda_{max}\{D(\sigma_t)\}$, $\underline{\lambda} = \lambda_{min}\{D(\sigma_t)\}$. For the $\mathcal{L}U_2(t)$, one can see

$$\mathcal{L}U_2(t) = e^{\top}(t)\mathbf{P}(\sigma_t)\hat{H}(e(t), \sigma_t) + ce^{\top}(t)(\mathbf{P}(\sigma_t) \otimes A(\sigma_t))e(t) - \rho e^{\top}(t)\mathbf{P}(\sigma_t)D(\sigma_t)e(t-\tau). \tag{13}$$

From the definition of $A(r)$, we know that $\lambda(A(r)) \le 0$, which yields

$$ce^{\top}(t)(\mathbf{P}(\sigma_t) \otimes A(\sigma_t))e(t) \le 0. \tag{14}$$

According to Assumption 2, we obtain

$$\begin{aligned} &e^{\top}(t)\mathbf{P}(\sigma_t)\hat{H}(e(t),\sigma_t) - \rho e^{\top}(t)\mathbf{P}(\sigma_t)D(\sigma_t)e(t-\tau) \\ \le& -\varphi e^{\top}(t)\mathbf{P}(\sigma_t)e(t) + e^{\top}(t)(I_N \otimes P(\sigma_t)\Delta)e(t) \\ \le& -\lambda_1 \|e(t)\|^2, \end{aligned} \tag{15}$$

where $\lambda_1 = \lambda_{min}[\varphi\mathbf{P}(\sigma_t) - I_N \otimes P(\sigma_t)\Delta]$. By substituting (14), (15) into (13), we then obtain

$$\mathcal{L}U_2(t) \le -\lambda_1 \|e(t)\|^2. \tag{16}$$

Furthermore, by Assumption 3, one can see

$$\mathcal{L}U_3(t) \le \frac{\bar{p}\omega}{2}\|e(t)\|^2 R(x(t),\sigma_t) - R(s(t),\sigma_t), \tag{17}$$

where $\omega = max\{\omega_\ell\}$. Based on the properties of the Markov process, we can obtain

$$\begin{aligned} \mathcal{L}U_4(t) =& \frac{1}{2}\sum_{l=1}^{m} q_{\sigma_t l} e^T(t)\mathbf{P}(l)e(t) \\ \le& \frac{1}{2}\sum_{l=1, l\neq\sigma_t}^{m} \bar{p} q_{\sigma_t l} e^T(t)e(t) + q_{ll}\underline{p}e^T(t)e(t) \\ \le& \pi\|e(t)\|^2, \end{aligned} \tag{18}$$

where $\pi = \frac{1}{2}\sum_{l=1, l\neq\sigma_t}^{m} \bar{p} q_{\sigma_t l} + \underline{p} q_{ll}$.

Step 3, substituting (12)-(18) into (11), we can get;

$$\mathcal{L}U(e(t),\sigma_t) \le -c\|e(t)\|^2, \tag{19}$$

where $c = [\rho\bar{p}\underline{\lambda} + \lambda_1 - \frac{\bar{p}\omega}{2} - \pi]$.

The proof of Lemma 1 is, therefore, completed.

**Lemma 2.**

Given that Assumptions (5) and (6) hold, the solution of the complex (4) satisfies

$$\sup_{-\tau\le t<\infty} E\|e(t)\|^q < \infty. \tag{20}$$

**Theorem 1.** Under Assumptions (1)-(6), if $U_x(e(t),\sigma_t)$ satisfies that $U_x(e(t),\sigma_t) \le \|e(t)\|^2$, the delay pinning feedback control and $d > 0$, the coupled network (1) can achieve $H_\infty$-synchronization.

Proof: Step 1: According to (19) and (10), one can see that

$$\mathcal{L}V(\hat{e}_t,\hat{\sigma}_t) \le -c\|e(t)\|^2 + I(t). \tag{21}$$

By Assumption 4, it is easy to see that

$$U_x(e(t),\sigma_t)[u(e(t-\tau),\sigma_t)-u(e(t),\sigma_t)] \le \frac{\beta}{2}U_x(e(t),\sigma_t)^2+\frac{1}{2\beta}|u(e(t-\tau),\sigma_t)-u(e(t),\sigma_t)|^2$$
$$\le \frac{\beta}{2}\|e(t)\|^2+\frac{\gamma^2}{2\beta}\|e(t)-e(t-\tau)\|^2. \tag{22}$$

According to the inequality $(a+b)^2 \le 2(a^2+b^2)$ and Assumption 1,

$$\begin{aligned}
&\frac{2\gamma^2}{\beta}\tau[\tau\|\hat{H}(e(t),\sigma_t)+cA(\sigma_t)e(t)-\rho D(\sigma_t)e(t-\tau)\|^2+\|\hat{R}(e(t),\sigma_t)\|^2]\\
\le&\frac{4\gamma^2\tau^2}{\beta}\|\hat{H}(e(t),\sigma_t)\|^2+\frac{4\gamma^2\tau^2\rho^2\overline{\lambda}^2}{\beta}\|e(t-\tau)\|^2+\frac{2\gamma^2\tau}{\beta}\|\hat{R}(e(t),\sigma_t)\|^2\\
\le&\frac{4\gamma^2\tau^2}{\beta}\|H(x(t),\sigma_t)-H(s(t),\sigma_t)\|^2+\frac{4\gamma^2\tau^2\rho^2\overline{\lambda}^2}{\beta}\|e(t-\tau)\|^2+\frac{2\gamma^2\tau}{\beta}\|R(x(t),\sigma_t)-R(s(t),\sigma_t)\|^2\\
\le&\frac{2\gamma^2}{\beta}(2\tau^2\eta+\tau\alpha)\|e(t)\|^2+\frac{4\gamma^2\tau^2\rho^2\overline{\lambda}^2}{\beta}\|e(t-\tau)\|^2.
\end{aligned}\tag{23}$$

where $\eta=\max\{\eta_1,\eta_2,\cdots,\eta_N\}$, $\alpha=\max\{\alpha_1,\alpha_2,\cdots,\alpha_N\}$,$\ell=1,2,\cdots,N$, $\eta_\ell$ and $\alpha_\ell$ are defined in Assumption 1. Noting $\tau \le \sqrt{\frac{\beta}{16\rho^2\overline{\lambda}^2}}$, we have

$$\frac{4\gamma^2\tau^2\rho^2\overline{\lambda}^2}{\beta^2}\|e(t-\tau)\|^2 \le \frac{8\gamma^2\tau^2\rho^2\overline{\lambda}^2}{\beta^2}\|e(t)\|^2+\frac{8\gamma^2\tau^2\rho^2\overline{\lambda}^2}{\beta^2}\|e(t)-e(t-\tau)\|^2$$
$$\le\frac{8\gamma^2\tau^2\rho^2\overline{\lambda}^2}{\beta^2}\|e(t)\|^2+\frac{\gamma^2}{2\beta}\|e(t)-e(t-\tau)\|^2. \tag{24}$$

Substituting (22), (23), (24) into (21), we can obtain

$$\mathcal{L}V(\hat{e}_t,\hat{\sigma}_t) < -\chi\|e(t)\|^2+\frac{\gamma^2}{\beta}\|e(t)-e(t-\tau)\|^2-\frac{2\gamma^2}{\beta}\int_{t-\tau}^{t}(\tau\|\hat{H}(e(v),\sigma_v)+cA(\sigma_v)e(v)$$
$$-\rho D(\sigma_v)e(v-\tau)\|^2+\|\hat{R}(e(v),\sigma_v)\|^2)dv, \tag{25}$$

where $\chi=[c-\frac{\beta\bar{p}^2}{2}-\frac{8\gamma^2\tau^2\rho^2\overline{\lambda}^2}{\beta^2}-\frac{2\gamma^2}{\beta}(2\tau^2\eta+\tau\alpha)]$.

Follows from the error system (4) that, for $t \ge \tau$

$$\begin{aligned}
&\|e(t)-e(t-\tau)\|^2\\
=&\left\|\int_{t-\tau}^{t}[\hat{H}(e(v),\sigma_v)+cA(\sigma_v)e(v)-\rho D(\sigma_v)e(v-\tau)]dv+\int_{t-\tau}^{t}\hat{R}(e(v),\sigma_v)dB(v)\right\|^2\\
\le&2\int_{t-\tau}^{t}(\tau\|\hat{H}(e(v),\sigma_v)+cA(\sigma_v)e(v)-\rho D(\sigma_v)e(v-\tau)\|^2+\|\hat{R}(e(v),\sigma_v)\|^2)dv.
\end{aligned}\tag{26}$$

Substituting (26) into (25), we can obtain

$$\mathcal{L}V(\hat{e}_t,\hat{\sigma}_t) < -\chi\|e(t)\|^2. \tag{27}$$

Step 2: Using a similar method in Theorems 3 and 4[23], we define the stopping time as follows

$$\zeta_k = inf\,\{t \ge 0 : \|e(t)\| \ge k\}. \tag{28}$$

According to Lemma 1, $U(e(t),\sigma_t)$ satisfied assumption 6, and $h(x,\ell)$ satisfied assumption 5; by lemma 2, $\zeta_k$ is increasing to infinity almost surely as $k \longrightarrow \infty$. Where $t \geq 0$, $H \in C^{2,1}(R^n \times S; R_+)$.
Then, we can obtain

$$EV(\hat{e}_{t\wedge\zeta_k}, \hat{\sigma}_{t\wedge\zeta_k}) \leq V(\hat{e}_0, \hat{\sigma}_0) + E\int_0^{t\wedge\zeta_k} \mathcal{L}V(\hat{e}_s, \hat{\sigma}_s)ds. \tag{29}$$

for any $t \geq 0$ and $k \geq k_0$, we can let $k \longrightarrow \infty$ and then apply the Fubini theorem to get

$$EV(\hat{e}_t, \hat{\sigma}_t) \leq V(\hat{e}_0, \hat{\sigma}_0) + \int_0^t E\mathcal{L}V(\hat{e}_s, \hat{\sigma}_s)ds, \tag{30}$$

for any $t \geq 0$, substituting (27) into (30), we obtain

$$EV(\hat{e}_t, \hat{\sigma}_t) \leq V(\hat{e}_0, \hat{\sigma}_0) - \chi\int_0^t E\|e(s)\|^2 ds. \tag{31}$$

We can get from (31)

$$\chi\int_0^t E\|e(s)\|^2 ds \leq V(\hat{e}_0, \hat{\sigma}_0). \tag{32}$$

Step 3: Noting that $\chi > 0$, we see from the above inequality that

$$\int_0^t E\|e(s)\|^2 ds \leq \frac{V(\hat{e}_0, \hat{\sigma}_0)}{\chi}. \tag{33}$$

Letting $t \longrightarrow \infty$, we obtain that

$$\int_0^\infty E\|e(s)\|^2 ds < +\infty. \tag{34}$$

The proof is, therefore, complete.

**Theorem 2.** Under Assumptions (1)-(6), the solution of the controlled network (4) for any given initial data, the controlled system (1) is asymptotical synchronization in mean square.

Proof: For any $0 \leq t_1 < t_2 < \infty$, according to Assumptions 4 and 5, we can apply the Itô formula to show

$$\begin{aligned} &|E\|e(t_2)\|^2 - E\|e(t_1)\|^2| \\ \leq &\int_{t_1}^{t_2} 2K \sup_{-\tau\leq t<\infty} E\|e(t)\| + 2K \sup_{-\tau\leq t<\infty} E\|e(t)\|^2 + 2\gamma \sup_{-\tau\leq t<\infty} E\|e(t)\|\|e(t-\tau)\| + K^2 \sup_{-\tau\leq t<\infty} (1+\|e(t)\|)^2 \\ \leq &\Theta(t_2 - t_1), \end{aligned} \tag{35}$$

where $\Theta$ is a constant independent of $t_1$ and $t_2$. That is, $\lim_{t\longrightarrow+\infty} E\|e(t)\|^2 = 0$.

**Lemma 3.**(Hanalay inequality) Let w(t) be a nonnegative function defined on the interval $[t_0 - \tau, \infty)$, and be continuous on the subinterval $[t_0, \infty)$. If there exist two positive constants $a, b$ satisfying $a > b$, such that:

$$\dot{w}(t) \leq -aw(t) + bw(t-\tau), \qquad t \geq t_0,$$

then $w(t) \leq w_{t_0} e^{-\gamma(t-t_0)}$, $t \geq t_0$, there $w_{t_0} = \sup_{t_0-\tau\leq t\leq t_0} w(t)$. $\gamma > 0$ is the smallest real root of the equation $a - \gamma - be^{\gamma\tau} = 0$.

**Theorem 3.** Under Assumptions (1)-(6), and the delay feedback pinning control given by (2). $\varepsilon\tau \leq \frac{1}{2}$ exists; the coupled network (1) can achieve exponential synchronization.

Proof: We choose Lyapunov function $V(\hat{e}_t, \hat{\sigma}_t)$ as defined by (8). Similar to the proof in Step 2 of Theorem 1 and according to the formula (30), we can show that

$$e^{\varepsilon t} EV(\hat{e}_t, \hat{\sigma}_t) \leq V(\hat{e}_0, \hat{\sigma}_0) + \int_0^t e^{\varepsilon s} E\Big(\varepsilon V(\hat{e}_s, \hat{\sigma}_s) + \mathcal{L}V(\hat{e}_s, \hat{\sigma}_s)\Big) ds. \tag{36}$$

Let $h_1 = \min_{\ell \in S} \underline{p}_\ell$, $h_2 = \max_{\ell \in S} \bar{p}_\ell$. From (8), we can get

$$V(\hat{e}_s, \hat{\sigma}_s) \leq \frac{h_2}{2} \|e(s)\|^2 + J_1, \tag{37}$$

where

$$J_1 = \frac{2\gamma^2}{\beta} \int_{-\tau}^0 \int_{t+s}^t [\tau \|\hat{H}(e(v), \sigma_v) + cA(\sigma_v)e(v) - \rho D(\sigma_v)e(v-\tau)\|^2 + \|\hat{R}(e(v), \sigma_v)\|^2] dv ds.$$

Similar to what we did in Theorem 1, we can show that:

$$\mathcal{L}V(\hat{e}_s, \hat{\sigma}_s) \leq -\Phi \|e(s)\|^2 + \frac{\gamma^2}{2\beta} \|e(s) - e(s-\tau)\|^2 + \frac{4\gamma^2\tau^2\rho^2\bar{\lambda}^2}{\beta^2} \|e(s-\tau)\|^2 - J_2, \tag{38}$$

where: $J_2 = \frac{2\gamma^2}{\beta} \int_{s-\tau}^s (\tau \|\hat{H}(e(v), \sigma_v) + cA(\sigma_v)e(v) - \rho D(\sigma_v)e(v-\tau)\|^2 + \|\hat{R}(e(v), \sigma_v)\|^2) dv$,$\Phi = [c - \frac{\beta\bar{p}^2}{2} - \frac{2\gamma^2}{\beta}(2\tau^2\eta + \tau\alpha)]$. For all $t \geq 0$, where $\varepsilon$ is a sufficiently small positive number to be determined later. Substituting (37) and (38) into (36), then we have

$$\begin{aligned}
e^{\varepsilon t} \frac{h_1}{2} E(\|e(t)\|^2) \leq& V(\hat{e}_0, \hat{\sigma}_0) + \int_0^t e^{\varepsilon s} E\Big(\varepsilon \frac{h_2}{2} \|e(s)\|^2 + \varepsilon J_1 - \Phi \|e(s)\|^2 + \frac{\gamma^2}{2\beta} \|e(s) - e(s-\tau)\|^2 \\
&+ \frac{4\gamma^2\tau^2\rho^2\bar{\lambda}^2}{\beta^2} \|e(s-\tau)\|^2 - J_2\Big) ds \\
\leq& V(\hat{e}_0, \hat{\sigma}_0) + \int_0^t e^{\varepsilon s} E\Big((\varepsilon \frac{h_2}{2} - \Phi) \|e(s)\|^2 + \frac{\gamma^2}{2\beta} \|e(s) - e(s-\tau)\|^2 \\
&+ \frac{4\gamma^2\tau^2\rho^2\bar{\lambda}^2}{\beta^2} \|e(s-\tau)\|^2 + \varepsilon J_1 - J_2\Big) ds.
\end{aligned} \tag{39}$$

Making use of (26), we can see:

$$\frac{\gamma^2}{2\beta} \|e(s) - e(s-\tau)\|^2 \leq \frac{1}{2} J_2. \tag{40}$$

According to the definitions of $J_1$ and $J_2$, we can get

$$J_1 \leq \tau J_2. \tag{41}$$

Substituting (41) and (40) into (39), we can get

$$\begin{aligned}
e^{\varepsilon t} \frac{h_1}{2} E(\|e(t)\|^2) \leq& V(\hat{e}_0, \hat{\sigma}_0) - (\Phi - \frac{\varepsilon h_2}{2}) \int_0^t e^{\varepsilon s} E(\|e(s)\|^2) ds + \frac{4\gamma^2\tau^2\rho^2\bar{\lambda}^2}{\beta^2} \int_0^t e^{\varepsilon s} E(\|e(s-\tau)\|^2) ds \\
&+ (\varepsilon\tau - \frac{1}{2}) \int_0^t e^{\varepsilon s} E(J_2) ds.
\end{aligned} \tag{42}$$

We can choose a sufficiently small $\varepsilon > 0$, such that:

$$\varepsilon\tau \leq \frac{1}{2} \quad , \quad \Phi - \frac{\varepsilon h_2}{2} > \frac{4\gamma^2\tau^2\rho^2\overline{\lambda}^2}{\beta^2}.$$

According to Lemma 3, we can then easily show that

$$E(\|e(t)\|^2) \leq Ge^{-\varepsilon t}. \tag{43}$$

Where: $G = \frac{2}{h_1}\left(V(\hat{e}_0, \hat{\sigma}_0) + \sup_{-\varepsilon \leq s \leq 0} E\|e(s)\|^2\right)$.

The proof is complete.

**Theorem 4.** Under the same condition of Theorem 3, the network (1) is almost surely exponentially synchronization.

Proof: Let $k$ be any nonnegative integer. According to Assumption 1, the Hölder inequality and the Doob martingale inequality, we can obtain that

$$E\left(\sup_{k \leq t \leq k+1} \|e(t)\|^2\right) \leq 3E\|e(k)\|^2 + 3\int_k^{k+1} E\left(\|\eta e(t) - \rho\underline{\lambda} e(t-\tau)\|^2\right)dt + 12\alpha^2\int_k^{k+1} E\|e(t)\|^2 dt.$$

By the inequality $(a+b)^2 \leq 2a^2 + 2b^2$, it is then to show that

$$E\left(\sup_{k \leq t \leq k+1} \|e(t)\|^2\right) \leq 3E\|e(k)\|^2 + (6\eta^2 + 12\alpha^2)\int_k^{k+1} E\|e(t)\|^2 dt + 6\rho^2\underline{\lambda}^2\int_k^{k+1} E\|e(t-\tau)\|^2 dt.$$

According to (43), we can get

$$\begin{aligned} E\left(\sup_{k \leq t \leq k+1} (\|e(t)\|^2\right) &\leq 3Ge^{-\varepsilon k} + G[6\eta^2 + 12\alpha^2]\int_k^{k+1} e^{-\varepsilon t}dt + 6G\rho^2\underline{\lambda}^2\int_k^{k+1} e^{-\varepsilon(t-\tau)}dt \\ &\leq 3Ge^{-\varepsilon k} + G[6\eta^2 + 12\alpha^2]e^{-\varepsilon k} + 6G\rho^2\underline{\lambda}^2 e^{\frac{1}{2}}e^{-\varepsilon k} \\ &= Ce^{-\varepsilon k}. \end{aligned}$$

where $C = 3G[1 + 2\eta^2 + 4\alpha^2 + 2\rho^2\underline{\lambda}^2 e^{\frac{1}{2}}]$. According to Chebyshev's inequality, consequently:

$$\sum_{k=0}^{\infty} P\left(\sup_{k \leq t \leq k+1} \|e(t)\| > e^{-0.25\varepsilon k}\right) \leq \sum_{k=0}^{\infty} Ce^{-0.5\varepsilon k} < \infty.$$

By Borel-Cantelli lemma, for almost all $\omega \in \Omega$, there is positive integer $k_0 = k_0(\omega)$ such that

$$\sup_{k \leq t \leq k+1} \|e(t)\| \leq e^{-0.25\varepsilon k}, \qquad k \geq k_0.$$

Then, for almost all $\omega \in \Omega$

$$\frac{1}{t}\log(\|e(t)\|) \leq -\frac{0.25\varepsilon k}{k+1}, \qquad t \in [k, k+1], k \geq k_0.$$

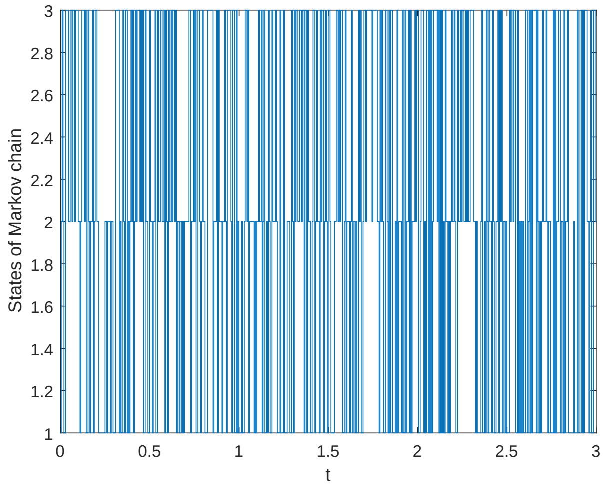


**Figure 1.** Time evolution of Markov chain $\{\sigma_t | t \in [0,3]\}$ that switches between the three states with generator $Q$.

This implies

$$\limsup_{t\to\infty} \frac{1}{t}\log(\|e(t)\|) \le -0.25\varepsilon < 0, \qquad a.s.$$

The proof is complete.

**Remark.** Our new result in this paper has removed the restrictive condition of existing research, which enables us to design a delay feedback control in order to stabilize a given unstable hybrid SDE. Furthermore, our new result can be used to achieve synchronization conditions for stochastic switched networks with Lévy noise.

## 4. EXAMPLE

Let us consider a linear n-dimensional unstable stochastic coupled network with delay pinning adaptive feedback control:

$$dx_i(t) = h(x_i(t),\sigma_t)dt + r(x_i(t),\sigma_t)dB(t) + c\sum_{j=1}^{5} a_{ij}(\sigma_t)x_j(t)dt + u(x_i(t-\tau),\sigma_t)dt \tag{44}$$

$$u(x_i(t-\tau),\sigma_t) = -\rho d_i(\sigma_t)(x_i(t-\tau) - s(t-\tau)) \tag{45}$$

where $x_i(t) = (x_i^1(t), x_i^2(t), x_i^3(t)) \in \mathbb{R}^2$; $d_i(\sigma_t) = I_{i\in\mathcal{D}(\sigma_t)}$ is the indication function for the pinned node subset $\mathcal{D}(\sigma_t) \subset \{1,2,3,4,5\}$. Let $c = 0.84$ and the control strength gain be $\rho = 8$. The desirable trajectory $s(t) = (s^1(t), s^2(t), s^3(t))^\top$ is described by (3).

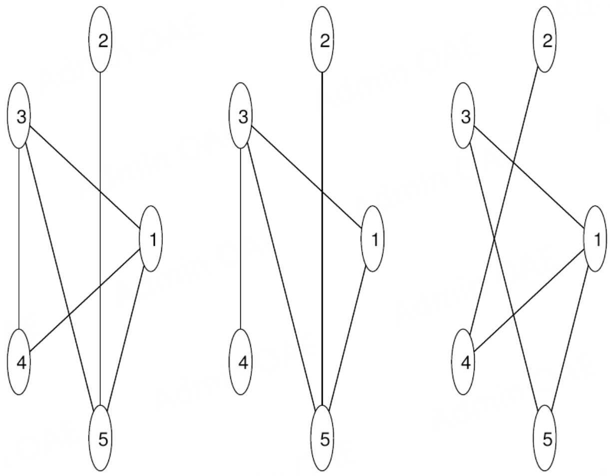


**Figure 2.** The topological structures $A_i\,(i=1,2,3)$ of the complex network.

The state space of the Markov chain $\sigma_t$ is $\mathbb{S}=\{1,2,3\}$ with generator as $Q=\begin{bmatrix}8 & -3 & -5\\ -2 & 5 & -3\\ -4 & -5 & 9\end{bmatrix}$. The time evolution of Markov chain $\sigma_t$ is depicted in Figure 1, showing the underlying switching. And the coupling matrix $A(\sigma_t)=(a_{ij}(\sigma_t))_{5\times 5}$ is switched according to $\sigma_t$ as follows: $\sigma_t=\ell$, then $A(\ell)=A_\ell$; one of possible topological structures is shown in Figure 2.

The coupling matrix corresponding to Figure 2 is shown as follows,

$$A_1=\begin{bmatrix}-3 & 0 & 1 & 1 & 1\\ 0 & -1 & 0 & 0 & 1\\ 1 & 0 & -3 & 1 & 1\\ 1 & 0 & 1 & -2 & 0\\ 1 & 1 & 1 & 0 & -3\end{bmatrix}, A_2=\begin{bmatrix}-2 & 0 & 1 & 0 & 1\\ 1 & -1 & 0 & 0 & 1\\ 1 & 0 & -3 & 1 & 1\\ 0 & 0 & 1 & -1 & 0\\ 1 & 1 & 1 & 0 & -3\end{bmatrix}, A_3=\begin{bmatrix}-3 & 0 & 1 & 1 & 1\\ 0 & -1 & 0 & 1 & 0\\ 1 & 0 & -2 & 0 & 1\\ 1 & 1 & 0 & -2 & 0\\ 1 & 0 & 1 & 0 & -2\end{bmatrix}$$

The noise is described by Brownian motion $B_i(t)$, which is given in Figure 3.

Consider a network of Chua's circuits. The individual node dynamics of Chua's circuit can be expressed as follows:

$$h(x_i(t),\sigma_t)=\begin{pmatrix}z_1(\sigma_t)(-x_1(t)+x_2(t)-h_1(x_1(t)))\\ x_1(t)-x_2(t)+x_3(t)\\ -z_2(\sigma_t)x_2(t)\end{pmatrix}$$

where $h_1(x)=\sigma_1 x+1/2(\sigma_2-r_1)(|x+1|-|x-1|), z_1(1)=9.78, z_1(2)=3.38, z_1(3)=9.98, z_2(1)=100.24, z_2(2)=30.24, z_2(3)=1.4, \sigma_1(1)=-0.3, \sigma_1(2)=-0.5, \sigma_1(3)=-2.35, \sigma_2(1)=-0.1, \sigma_2(2)=-1.6, \sigma_2(3)=0.6$.
$r(x_i(t),\sigma_1)=tanh(x), r(x_i(t),\sigma_2)=tanh(x)-1, r(x_i(t),\sigma_3)=tanh(x)-2$. In order to make $h(x_i(t),\sigma_t)$ and $r(x_i(t),\sigma_t)$ satisfies Assumption 1, we choose $\eta=3.421$. $\alpha=2.854$.

Let $P=I_3, \Delta=12.6I_3$ and $\varphi=2.9846, \gamma=0.2437, \beta=0.427, \overline{\lambda}=0.143, \bar{p}=1.06, \rho=12.4, h_2=8.37$, Then, $\tilde{H}(x(t),\sigma_t)$ satisfied Assumption 2; $u(x,i)$ satisfied Assumption 4.

Considering the intensity functions $R(\cdot)$, we select $R(x_i(t),\sigma_t)=0.1\cdot\sigma_t\cdot \mathrm{diag}\{x_i^1(t),x_i^2(t),x_i^3(t)\}$. Then, we can get $\mathrm{trace}(R^TR)\leq 0.03e_i^T(t)e_i(t)$. Let $\omega=0.03$, then Assumption 3 holds

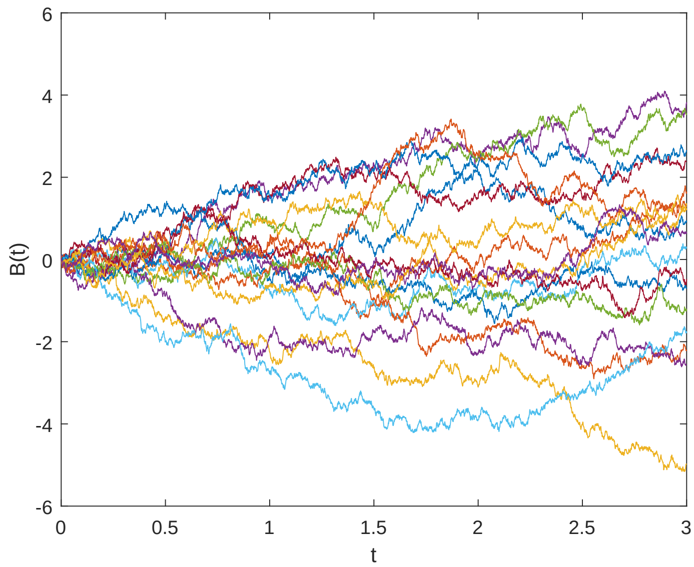


**Figure 3.** Time evolution of Brownian motion $B(t)$.

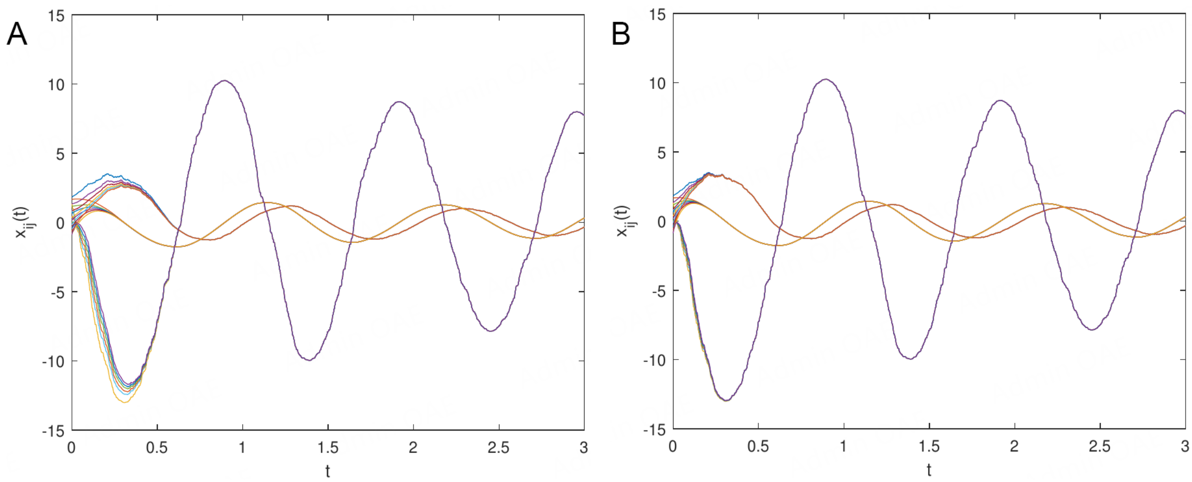


**Figure 4.** Shows that asymptotical synchronization in mean square(A) and almost surely exponentially synchronization(B) of asymptotical synchronization in mean square and almost surely exponentially synchronization of the State trajectories $x_i^j(t)(i = 1, 2, \cdots, 5, j = 1, 2, 3)$ for networks system (44) under delay pinning feedback control in [0,3] separately.

According to Theorem 1, we can see $\tau \leq \sqrt{\frac{\beta}{16\rho^2\overline{\lambda}^2}} \approx 0.3366$. From Theorem 3, $\varepsilon\tau \leq \frac{1}{2}$, then $\varepsilon \leq \frac{1}{2\tau} \approx 1.4854$. Then, it follows that $\Phi = [c - \frac{\beta\bar{p}^2}{2} - \frac{2\gamma^2}{\beta}(2\tau^2\eta + \tau\alpha)] \approx 0.1337$ and $\Phi - \frac{\varepsilon h_2}{2} > \frac{4\gamma^2\tau^2\rho^2\overline{\lambda}^2}{\beta^2}$.

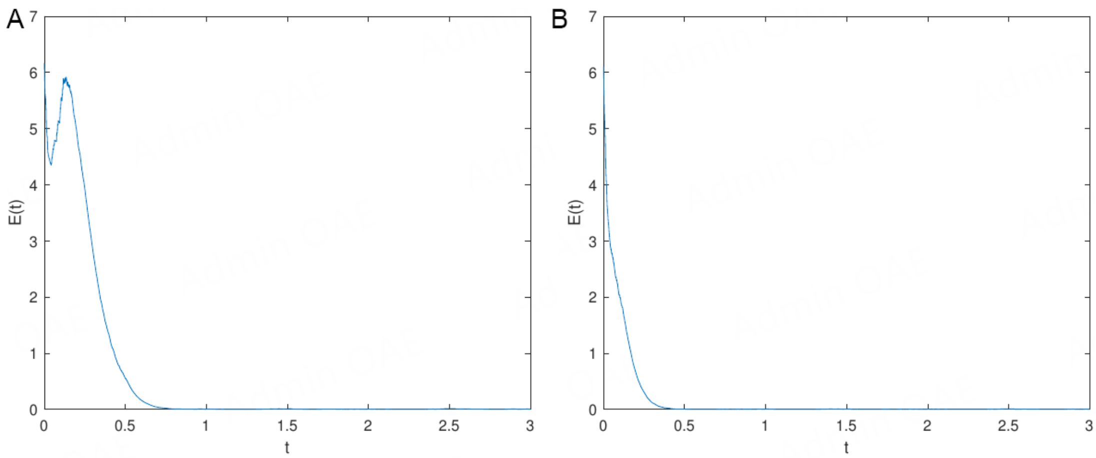


**Figure 5.** The trajectory of synchronization error $E(t)$ of asymptotical synchronization in mean square (A) and almost surely exponentially synchronization (B) networks system (44) under delay pinning feedback control in [0,3] separately.

As demonstrated by Figure 4, Figure 4A shows that the State trajectories for networks system (44) under Theorem 2; Figure 4B presents that the State trajectories for networks system (44) under Theorem 4; the state variables of all nodes of the Markov switched stochastic complex networks can achieve the synchronization in a very short time.

Denote the total synchronization error $E(t)$ by $E(t) = \sqrt{\sum_{i=1}^{5}\sum_{j=1}^{3}(x_i^j(t) - s_j(t))^2}$.

Figure 5 demonstrates that the total error converges to zero after a very short time. Figure 5A illustrates that all nodes in the stochastic complex network (44) achieve the asymptotical synchronization in mean square, which also indicates the convergent efficiency under the proposed framework. Figure 5B shows the complex network (44) achieves the almost surely exponentially synchronization; $E(t)$ converges to zero after a very short time.

## 5. CONCLUSION

By using a time-delay feedback controller that depends on the past state, we study a class of Stochastic Switched Networks with Markov switching and Brown noise in this paper. We obtain the sufficient condition of asymptotical synchronization in mean square, $H_\infty$-synchronization and almost sure exponential synchronization in our framework. The main method includes inequality techniques, Itô formula and Borel-Cantelli Lemma. Finally, we illustrate our theory with an example of simulation. Our future effort will focus on the asymptotical synchronization in mean square, $H_\infty$-synchronization and almost sure exponential synchronization of highly nonlinear Markov switched stochastic network. If Assumptions (1) - (6) do not hold, the delay feedback control method in this article will be inapplicable and new methods need to be introduced for exploration. General feedback control may need to be set up, which will be our next consideration.

## DECLARATIONS

### Authors' contributions

Writing the manuscript: Li Z, Tang J (Juan Tang), Cheng F, Dong H
Organizing the manuscript: Li Z, Cheng F, Tang J (Juan Tang), Dong H, Tang J (Jianliang Tang)
Experimental data presented in the manuscript: Cheng F, Tang J (Juan Tang), Li Z
Discussion of the manuscript: Dong H, Cheng F, Li Z, Tang J (Juan Tang), Tang J (Jianliang Tang)

### Availability of data and materials

Not applicable.

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**Conflicts of interest**
All authors declared that there are no conflicts of interest.

**Ethical approval and consent to participate**
Not applicable.

**Consent for publication**
Not applicable.



## REFERENCES

1. Ahmed HM, Ahmed AMS, Ragusa MA. On some non-instantaneous impulsive differential equations with fractional brownian motion and Poisson jumps. *TWMS J Pure Appl Math* 2023;14:125-40. DOI
2. Al-Askar FM. Impact of fractional derivative and brownian motion on the solutions of the Radhakrishnan-Kundu-Lakshmanan equation. *J Funct Space* 2023;2023:8721106. DOI
3. Benkabdi Y, Lakhel EH. Exponential stability of delayed neutral impulsive stochastic integro-differential systems perturbed by fractional Brownian motion and Poisson jumps. *Filomat* 2023;37:8829-44. DOI
4. Ghosh D, Frasca M, Rizzo A, et al. The synchronized dynamics of time-varying networks. *Phys Rep* 2022;949:1-63. DOI
5. Li M, Liu RR, Lü L, Hu MB, Xu S, Zhang YC. Percolation on complex networks: theory and application. *Phys Rep* 2021;907:1-68. DOI
6. Dutta S, Khanna A, Assoa AS, et al. An ising hamiltonian solver based on coupled stochastic phase-transition nano-oscillators. *Nat Electron* 2021;4:502-12. DOI
7. Somers VLJ, Manchester IR. Sparse resource allocation for spreading processes on temporal-switching networks. *IFAC-PapersOnLine* 2023;56:7387-93. DOI
8. Zhong J, Ho DWC, Lu J. A new approach to pinning control of Boolean networks. *IEEE Trans Control Network Syst* 2021;9:415-26. DOI
9. Zhou W, Zhu Q, Shi P, Su H, Fang J, Zhou L. Adaptive synchronization for neutral-type neural networks with stochastic perturbation and Markovian switching parameters. *IEEE Trans Cybern* 2014;44:2848-60. DOI
10. Giap VN, Nguyen QD, Huang SC. Synthetic adaptive fuzzy disturbance observer and sliding-mode control for chaos-based secure communication systems. *IEEE Access* 2021;9:23907-28. DOI
11. Dong H, Luo M, Xiao M. Synchronization for stochastic coupled networks with Lévy noise via event-triggered control. *Neural Netw* 2021;141:40-51. DOI
12. Wu Y, Shen B, Ahn CK, Li W. Intermittent dynamic event-triggered control for synchronization of stochastic complex networks. *IEEE Trans Circuits Syst I* 2021;68:2639-50. DOI
13. Gambuzza LV, Di Patti F, Gallo L, et al. Stability of synchronization in simplicial complexes. *Nat Commun* 2021;12:1255. DOI
14. Lin H, Wang C, Chen C, et al. Neural bursting and synchronization emulated by neural networks and circuits. *IEEE Trans Circuits Syst I* 2021;68:3397-410. DOI
15. Jo J, Lee S, Hwang SJ. Score-based generative modeling of graphs via the system of stochastic differential equations. *Int Conf Mach Learn* 2022;10362-83. DOI
16. Luo D, Tian M, Zhu Q. Some results on finite-time stability of stochastic fractional-order delay differential equations. *Chaos Solitons Fractals* 2022;158:111996. DOI
17. Vaseghi B, Hashemi SS, Mobayen S, Fekih A. Finite time chaos synchronization in time-delay channel and its application to satellite image encryption in OFDM communication systems. *IEEE Access* 2021;9:21332-44. DOI
18. Zhou L, Zhu Q, Wang Z, Zhou W, Su H. Adaptive exponential synchronization of multislave time-delayed recurrent neural networks with levy noise and regime switching. *IEEE Trans Neural Netw Learn Syst* 2017;28:2885-98. DOI
19. Qi W, Hou Y, Zong G, Ahn CK. Finite-time event-triggered control for semi-Markovian switching cyber-physical systems with FDI attacks and applications. *IEEE Trans Circuits Syst I* 2021;68:2665-74. DOI
20. Lu Z, Hu J, Mao X. Stabilisation by delay feedback control for highly nonlinear hybrid stochastic differential equations. *Discrete Cont Dyn Syst* 2019;24:4099-116. DOI
21. Liptser RS, Shiryayev AN. Theory of martingales. Kluwer Academic Publishers; 1982, p. 49. DOI
22. Mao X. Stochastic differential equations and applications. Elsevier; 2008. Available from: https://www.sciencedirect.com/book/9781904275343/stochastic-differential-equations-and-applications [Last accessed on 25 Jun 2024]

23. Fei W, Hu L, Mao X, Shen M. Delay dependent stability of highly nonlinear hybrid stochastic systems. *Automatica* 2017;82:65-70. DOI