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Region stability of switched two-dimensional linear dissipative Hamiltonian systems with multiple equilibria

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Abstract

This paper studies the issues of region stability of switched two-dimensional linear dissipative Hamiltonian systems. Such switched systems are composed of two stable subsystems with two different equilibrium points. Since the equilibrium points of two subsystems are different, and the state matrices of subsystems may not commute, it is difficult to address such switched systems. This paper considers the case that the switching path corresponding to the switched systems is a switching line passing through the equilibrium points of two different subsystems. A suitable region containing all the equilibrium points of subsystems is first determined. Based on the concept of region stability of switched systems with multiple equilibrium points, this paper proposes some sufficient conditions of region stability and asymptotically region stability for such kind of switched linear dissipative Hamiltonian systems via the maximum energy function method. The above main results obtained can be applied to some classes of electronic circuits, such as switching DC/DC converters and AC/DC converters. As an application and illustration, a switching DC circuit and two numerical examples are carried out to show the effectiveness of the region stability results obtained in this paper.

Keywords: Switched linear systems, dissipative Hamiltonian systems, switched line, regional stability, asymptotic regional stability, multiple equilibrium points, maximum energy function method

1. INTRODUCTION

Hybrid systems are a kind of complex systems owning both continuous-time states and discrete time states. As an important class of hybrid systems, a switched system is composed of finite/infinite subsystems and a switching path/signal, which is used to select one subsystem to activate at any instant. Since 1990, it has been found that switched systems have been widely used in many physical and practical fields, such as power systems [1], multi-agent technology [2–5], hybrid electric vehicles [6], traffic control systems [7], and so on. Owing to the structural complexity and extensive application of switched systems, it is of great significance to study switched systems. In 1999, Liberzon and Morse published a review on the issues of stability analysis of switched systems [8]. The review summarized that there are three basic problems with the stability of the switched system as follows. One is the stability of switched systems under any switching paths; Another is the stability of the system under given switching paths; The third is to determine suitable switching paths such that switched systems are stable under the determined switching paths. After that, a large number of researchers have paid increasing attention to the domain of switched systems and proposed plenty of results for switched systems. Meanwhile, some important methods are developed to analyze the stability of switched systems, such as the common Lyapunov function method (CLF) [9], the multi-Lyapunov functions method (MLF) [10], the multi-storage functions method (MSFs) [11], and so on.

Almost all of the above system stability analysis and comprehensive results obtained for switched systems are based on the same assumption that all subsystems have a common single equilibrium point, i.e., the origin, in their common state domain [12–16]. However, due to the complexity of the environment and system structures, for each subsystem of switched systems, there will be one or more equilibria in its state domain. Moreover, the equilibrium points of each subsystem in many switched systems may not be the same one. Since there is short of necessary stability analysis methods/techniques, it is indeed a challenge and has a practical significance to investigate switched systems with multiple equilibria (ME). Although it is more difficult to study switched systems with ME than switched systems with a common single equilibrium point, many researchers have presented some stability results in the literature. For example, in the reference [17], several sufficient conditions of region stability, global asymptotic region stability, and region instability are proposed for switched linear time-invariant systems under arbitrary periodical/quasi-periodical switching paths with respect to a region. The corresponding region contains all the multiple equilibrium points of such kind of switched linear systems with ME. The results of the boundedness and practical stability are obtained for switched systems with ME in the literature [18] by studying the robustness to external disturbances of such switched systems.

As an effective method, the Hamiltonian function method has been widely used to study nonlinear systems. Such a method should be based on system energy control and system stability analysis. In practice, a general affine nonlinear system can be converted into a port-controlled Hamiltonian (PCH) system based on the Hamiltonian realization method [19]. The PCH systems can be accepted as a form of unified mathematical structures for various physical systems, such as electric power systems and mechanical systems. The Hamiltonian function of such a PCH system has an explicitly physical significance and has often been used as the total energy of physical systems. It is then often selected as an appropriate Lyapunov function. Based on the above advantages of Hamiltonian systems, researchers obtain lots of results of Hamiltonian systems, such as the results of trajectory tracking control [20,21], $\mathcal{H}_\infty$ control [22], etc.

Recently, as an important kind of switched systems, switched Hamiltonian systems (SHSs) have also been studied. This may be because such kind of switched systems can be used to model many practical systems composed of finite difference modes. In fact, studying the SHS may induce to provide an effective method for analyzing such a class of switched systems. However, compared to the existing vast results of switched systems with subsystems not being the formation of Hamiltonian systems, there are only a few results presented for SHSs in the open literature. Except for the literature [23] studying the stability issues of switched linear Hamiltonian systems, some other studies [24,25] analyze the issues of system stability, system stabilization, and

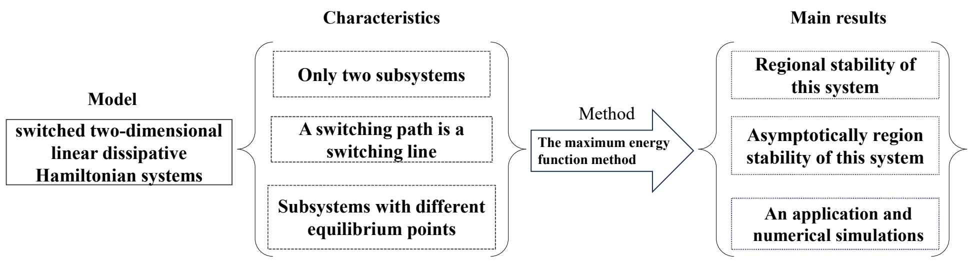

Figure 1. The framework of the article.

$\mathcal{H}_\infty$ control of the switched PCH systems and propose some corresponding results for such type of switched systems.

It should be pointed out that all the aforementioned results obtained for switched PCH systems are also based on the assumption that all subsystems must have a common single equilibrium point–the origin of the system state space. To the best knowledge of the authors, there are fewer results reported for SHSs with ME in the open literature, including the following one. The literature[26] tries to model a power system with a series of faults in the form of switched impulsive Hamiltonian systems (SIHSs) with ME and proposes some necessary and sufficient conditions of RS and ARS for the power system with respect to the region via the maximum energy function method, which was first introduced in the literature[15]. It is especially witnessed that there do not exist any results of switched linear Hamiltonian systems with ME in the literature. Therefore, studying switched linear Hamiltonian systems with multiple equilibrium points not only enriches the theoretical results of system analysis and system control of switched systems but also has important practical significance.

This paper studies the region stability issues of switched linear PCH systems with multiple equilibrium points. Due to the complexity involved in studying such a switched system composed of subsystems having multiple equilibrium points and non-commutative subsystems' state matrices, we exclusively consider three specific cases of the switched linear Hamiltonian system. One is the case that there are only two subsystems. Another is the case that each subsystem has only a unique equilibrium point, and the equilibrium points of the two subsystems are different. The third is the case that the involved switching path is the straight line passing through the equilibrium points of the two subsystems. To address these cases, we utilize the concepts of region stability and asymptotic region stability of switched systems with multiple equilibrium points. Furthermore, by means of the maximum energy method, we propose the main contributions of this paper as follows: (1) several sufficient conditions of the region stability and asymptotic region stability are given for switched linear PCH systems with respect to a region containing all multiple equilibrium points under the specific switching path; (2) an application of switching DC electric circuits and two numerical examples are carried out to illustrate the effectiveness and practicality of the theoretical results obtained in this paper. Figure 1 shows the framework of the content of this paper. Compared to the existing region stability results proposed in the literature[17], the region stability criteria obtained in this paper have the following significant advantages: (1) they are suitable to the case that any pairs of all the state matrices of subsystems do not commute between each other. However, for the sufficient conditions of the region stability results given in the literature[17], all the state matrices of subsystems of switched linear systems with ME are assumed to be commutative matrices; (2) dissimilar to that of region stability proposed in the literature[17], the sufficient conditions of region stability obtained in this paper do not require any information on the dwell-times of any subsystems; (3) the sufficient conditions of region stability presented in this paper are very easy to check whether the given switched linear Hamiltonian systems with switching lines are (asymptotically) region stable or not.

The rest of this paper is organized as follows. Section 2 gives the system expression of switched linear Hamiltonian systems with multiple equilibrium subsystems, definitions, and other preliminaries, including notations. Section 3 proposes the main contributions of this paper, i.e., some region stability criteria of switched linear Hamiltonian systems with multiple equilibrium points. Section 4 illustrates numerical examples and an application for switching DC circuits to show the validity of the obtained new results, which is followed by the conclusion in Section 5.

Notation: $\mathbb{R}$ and $\mathbb{R}_+$ denote the real number field and the positive real number field, respectively; $\mathbb{R}^l$ denotes the l-dimensional Euclidean space; $\|\cdot\|$ denotes the norm in the l-dimensional Euclidean space $\mathbb{R}^l$; $\mathbb{R}^{l\times s}$ denotes the set of $l \times s$ real matrices, $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{N}_+$ denotes the set of positive integers; $d(x, y) = \|x - y\|$, $d(x, \Omega) = \inf_{y\in\Omega} \|x - y\|$ denote the distances of the two points x and y contained in $\mathbb{R}^l$, the point $x \in \mathbb{R}^l$ to the compact domain $\Omega \subset \mathbb{R}^l$, respectively. For a given matrix J, J^T is the matrix transpose of J, $J = -J^T$ signifies that J is skew-symmetric. For a given matrix R, $R > 0$ and $R \geqslant 0$ represent that R is positive definite and positive semi-definite, respectively. $\nabla H(x) = \frac{\partial H(x)}{\partial x}$ denotes the gradient of the differentiable function $H(x)$. Let $\lambda_{ij}(A_i)$ be the eigenvalues of the matrix $A_i \in \mathbb{R}^{n\times n}$, where $i = 1, \dots, N$, $j = 1, \dots, n$, and $N \in \mathbb{N}$. And $\lambda_{\min}(A)$, $\lambda_{\max}(A)$ represent the smallest and the largest of all the eigenvalues of the positive definite matrix A, respectively.

2. PRELIMINARIES

This section gives the preliminaries needed necessarily for studying switched linear PCH systems with multiple equilibrium points in the next sections. Section 2.1 introduces the switched systems model considered in this paper and some preparatory knowledge and notation. Section 2.2 introduces some relevant definitions and a proposition that will be used in the sequel.

2.1. System description and preliminaries

Consider a switched linear Hamiltonian system with multiple equilibrium subsystems as follows.

$$\begin{cases} \dot{x} = \left[J_{\sigma(t)} - R_{\sigma(t)}\right] \nabla H_{\sigma(t)}(x), \\ x(t_0) = x_0, \end{cases} \quad \text{for all } t \geqslant t_0, \tag{1}$$

where $x = [x_1, x_2]^T \in \mathbb{D}$ represents the state of the system, and $\mathbb{D}$ is the common domain of all the subsystems of system (1). The map $\sigma(t)\colon [t_0, +\infty) \to \Lambda := \{1, 2\}$ is a piecewise right continuous constant step function, which is called to be a switching path or a switching rule. The value of the function $\sigma(\mathrm{t}) = i$, where $i = 1, 2$, means that the subsystem i is activated at the time instant t. The matrices $J_i \in \mathbb{R}^{2\times 2}$ and $R_i \in \mathbb{R}^{2\times 2}$ are all constant matrices. Moreover, $J_i^T = -J_i \neq 0$ and $R_i^T = R_i$, $i = 1, 2$. The notation $\nabla H_i(x) = \frac{\partial H_i(x)}{\partial x}$ is the gradient of the energy function $H_i(x)$ of subsystem i. The function $H_i(x)$, where $i = 1, 2$, satisfies the following two: (1) they are continuous and differentiable; (2) $H_i(x) > 0$ for any $x \in \mathbb{D} - \{x^{e1}, x^{e2}\}$, in which x^{ei} satisfying $H_i(x^{ei}) = 0$ is a unique equilibrium point of subsystem i. Moreover, the equilibrium points x^{e1} and x^{e2} of the two subsystems are different. In fact, system (1) considered in this paper is a switched two-dimensional linear system only consisting of two subsystems.

Remark 1 *Note that due to the special dissipative Hamiltonian structures of subsystems of system (1), i.e., the matrices $J_i^T = -J_i$ and $R_i^T = R_i \geqslant 0$, if the matrix $J_i \neq 0$, then the directions of trajectories of subsystems of system (1) are all revolved around the corresponding equilibrium points of subsystems anticlockwise or clockwise.*

2.1.1. Switching line

To facilitate the subsequent analysis, the switching path $\sigma(t)$ of system (1) can be expressed as $\sigma(t) = i_m \in \{1, 2\}$, $t \in [t_m, t_{m+1})$, $m = 0,1 \cdots$, and then we express by $x(t) := x(t; t_0, x_0, \sigma)$ the trajectory of system (1) under the switching path $\sigma(t)$ starting from the initial state $x(t_0)$ at the initial time t_0. Express by $\{x_m\}_{m=0}^{+\infty}$ the switching state sequence and by $\{t_m\}_{m=0}^{+\infty}$ the switching time sequence. Express by $\left\{H_{i_m}(x_m)\right\}_{m=0}^{+\infty}$ the switching energy sequence and by $\{i_m\}_{m=0}^{+\infty}$ the switching index sequence, where $x_m = x(t_m)$, $i_m \neq i_{m+1}$, $\forall m \in \mathbb{N}$. Express by t_{m_l}, $\forall l \in \mathbb{N}_+$, the switching time when the l-th subsystem is switched on.

Throughout this paper, the switching path $\sigma(t)$ governed system (1) is assumed to satisfy the following four cases:

A1. Switching phenomena of system (1) only appear on a straight line that passes through the two equilibrium points of the first subsystem and the second subsystem of system (1). Such a straight line is called a *switching line* in this paper and is denoted by l_1.

A2. Any one of all the subsystems cannot be excluded from those activated subsystems as time goes to infinity.

A3. The switching times of the switching path $\sigma(t)$ is a finite number over any finite time intervals.

A4. The whole state trajectory $x(t)$ of system (1) is continuous at any switching time instants t_m, where any $m \in \mathbb{N}$.

Remark 2 *It should be pointed out from Remark 1 that the trajectories of every subsystem of system* (1) *must pass through the switching line intercepting the equilibrium points x^{e1} and x^{e2} of the two different subsystems infinite times as time goes to infinity. Therefore, switching only on such a switching line can ensure that the switching is infinite as time goes to infinity.*

Remark 3 *The explanations and motivations of the above four assumptions are addressed as follows. As is well-known, there are many event-driven practical/physical dynamical systems that can be modeled by switched systems with special switching paths/strategies. Among these switching paths/ strategies caused by event-driven, there is a kind of switching path–switching lines. This is the motivation for Assumption A1. Since we just consider the final tendency of the trajectory of system* (1) *in this paper, it is not necessary to consider those subsystems that are not activated anymore after a certain finite time. Therefore, all the subsystems of* (1) *must be often activated as time goes to infinity, i.e., Assumption A2. The aim of Assumption A3 is to exclude the Zeno phenomena–the chatting, i.e., infinite switching occurs in any finite time interval. Assumption A4 is used to avoid the jump phenomenon of the state of system* (1).

Moreover, we let $x^{e1} = [x_1^{e1}, x_2^{e1}]^T$ and $x^{e2} = [x_1^{e2}, x_2^{e2}]^T$ be the equilibrium points of the first subsystem and the second subsystem, respectively. The state or trajectory of system (1) is denoted by $x(t) = [x_1, x_2]^T$. The initial state of the system (1) at the initial time t_0 is denoted by $x(t_0) = [x_1^0, x_2^0]^T$. Then, for the two equilibrium points x^{e1} and x^{e2} of the first and second subsystems, there exists a straight line passing through them. The straight line is denoted by l_1 in this paper. Without loss of generality, for the straight line l_1, there are two cases as follows.

(1) If the equilibrium points x^{e1} and x^{e2} of the two subsystems satisfying $x_1^{e2} > x_1^{e1}$, then the switching line l_1 can be expressed as

$$l_1 : x_2 = \frac{x_2^{e2} - x_2^{e1}}{x_1^{e2} - x_1^{e1}} x_1 + \frac{x_1^{e2} x_2^{e1} - x_2^{e2} x_1^{e1}}{x_1^{e2} - x_1^{e1}} = kx_1 + b, \tag{2}$$

where the two parameters of k and b are, respectively, as follows.

$$k = \frac{x_2^{e2} - x_2^{e1}}{x_1^{e2} - x_1^{e1}} \text{ and } b = \frac{x_1^{e2} x_2^{e1} - x_2^{e2} x_1^{e1}}{x_1^{e2} - x_1^{e1}}. \tag{3}$$

(2) If the equilibrium points x^{e1} and x^{e2} of the two subsystems satisfying the following: $x_1^{e1} = x_1^{e2}$, i.e., the slope of the line $k = \infty$, then the switching line l_1 can be expressed

$$l_1 : x_1 = x_1^{e1} = x_1^{e2}. \tag{4}$$

Remark 4 *Since the switching line l_1 in (2) passes through the equilibrium points x^{e1} and x^{e2} of the two different subsystems and the two different points are both determined in advance, both the slope k and the intercept b in (3) of the straight line l_1 can be easily obtained via the general method of analytic geometry.*

Based on the above, we know that the trajectory $x(t)$ of system (1) under the switching line l_1 passes into and out of the switching line l_1 as time goes to infinity. This implies that all the switching state sequences $\{x_m\}_{m=0}^{+\infty}$ are situated in the switching line l_1. Then, all the switching energy sequences $\{H_{i_m}(x_m)\}_{m=0}^{+\infty}$ and the switching time sequences $\{t_m\}_{m=0}^{+\infty}$ corresponding to the switching state sequence $\{x_m\}_{m=0}^{+\infty}$ are also related to the switching line l_1.

2.1.2. The Hamiltonian functions

In this paper, the Hamiltonian functions of the two subsystems of system (1) are assumed to be the following quadratic forms:

$$H_i(x) = \frac{1}{2}(x - x^{ei})^T Q_i (x - x^{ei}), \ i = 1, 2, \tag{5}$$

where Q_1 and Q_2 are two positive definite matrices.

Since the switching states x_m for any $m \in \mathbb{N}$ are all on the switching line l_1, we know from (5) that all the switching states satisfy the following formula:

$$L_i(x_1) = g_i(x_1 - x_1^{ei})^2, \ i = 1, 2, \tag{6}$$

where Q_i and $i = 1, 2$ are the same positive definite matrices in (5), and

$$g_i := \frac{1}{2}[1 \ k] Q_i \begin{bmatrix} 1 \\ k \end{bmatrix} > 0, \ i = 1, \ 2, \tag{7}$$

where k is defined as in (3).

For the case that $k = \infty$, one knows from the Hamiltonian functions expressed in (5) that all the switching states x_m for any $m \in \mathbb{N}$ satisfy the following formula

$$M_i(x_2) = ð_i(x_2 - x_2^{ei})^2, \tag{8}$$

where Q_i and $i = 1, 2$ are the same two positive definite matrices in (5), and

$$ð_i := \frac{1}{2}[0 \ 1] Q_i \begin{bmatrix} 0 \\ 1 \end{bmatrix} > 0. \tag{9}$$

2.2. Some definitions and propositions

This subsection refers to some definitions and gives a proposition that is needed to analyze the region stability of system (1) in the next section below.

Definition 1 *(The maximum energy function)*[15]*. The following function $H(x)$ is called the maximum energy function of system* (1)

$$H(x) := \max\{H_1(x), \ H_2(x)\}, \ \text{for all } x \in \mathbb{D} \tag{10}$$

Definition 2 *(The maximum switching energy sequence)*[15]*. The following sequence is known as the maximum switching energy sequence of the switching path $\sigma(t)$*

$$\left\{H(x_m)\right\}_{m=0}^{+\infty} := \left\{\max\left\{H_1(x_m),\ H_2(x_m)\right\}\right\}_{m=0}^{+\infty}. \tag{11}$$

Proposition 1 *Consider system (1). The Hamiltonian functions $H_1(x)$ and $H_2(x)$ of the subsystems satisfy the following formula*

$$\frac{1}{2}\lambda_{\min}(Q_i)d^2(x,x^{ei}) \leqslant H_i(x) \leqslant \frac{1}{2}\lambda_{\max}(Q_i)d^2(x,x^{ei}),\ \forall x \in \mathbb{D} - \{x^{ei}\},\ i = 1,\ 2, \tag{12}$$

where $\lambda_{\min}(Q_i)$ and $\lambda_{\max}(Q_i)$ are the minimum and maximum eigenvalues, respectively, of the matrix Q_i for $i = 1,\ 2$.

Proof: Letting

$$x^i := x - x^{ei},\ x \in \mathbb{R}^2,\ i = 1,\ 2, \tag{13}$$

We obtain from the Hamiltonian function $H_i(x)$ of the i-th subsystem expressed as in (5) that

$$H_i(x) = \frac{1}{2}(x^i)^T Q_i x^i =: H_i(x^i),\ i = 1,\ 2. \tag{14}$$

Since every Q_i is a real symmetric positive definite matrix, there is an orthogonal matrix P_i satisfying

$$P_i^T P_i = P_i P_i^T = E_i,\ i = 1,\ 2, \tag{15}$$

where E_i is an identity matrix, and an orthogonal transformation

$$x^i = P_i y^i,\ i = 1,\ 2, \tag{16}$$

such that the following two hold.

(A.) By means of the orthogonal matrix P_i, the positive definite matrix Q_i can be diagonalized into a diagonal matrix as follows.

$$P_i^T Q_i P_i = \Lambda_i := \begin{bmatrix} \lambda_{i1} & 0 \\ 0 & \lambda_{i2} \end{bmatrix} > 0, \tag{17}$$

where λ_{i1} and λ_{i2} are the two real positive eigenvalues of the positive definite matrix Q_i, where $i = 1,\ 2$.

(B.) The Hamiltonian function $H_i(x^i)$ in (14) can be transformed by the orthogonal transformation (16) into the following standard form:

$$H_i(x^i) = \frac{1}{2}(y^i)^T P_i{}^T Q_i P_i y^i = \frac{1}{2}(y^i)^T \Lambda_i y^i =: H_i(y^i),\ i = 1,\ 2, \tag{18}$$

where y^i is as follows.

$$y^i = \begin{bmatrix} y_i^1 \\ y_i^2 \end{bmatrix} = P_i^T x^i = P_i^T (x - x^{ei}),\ i = 1,\ 2. \tag{19}$$

It can be obtained from (13), (19), (16), and (15) that the square norm of the vector x^i is as follow.

$$d^2(x,x^{ei}) = \|x - x^{ei}\|^2 = \|x^i\|^2 = (x^i)^T x^i = (P_i y^i)^T (P_i y^i) = (y^i)^T y^i = \|y^i\|^2,\ i = 1,\ 2. \tag{20}$$

We know from (17)-(20) that the following two hold.

$$H_i(y^i) = \frac{1}{2}(y^i)^T \Lambda_i y^i = \frac{1}{2}\left[\lambda_{i1}(y_i^1)^2 + \lambda_{i2}(y_i^2)^2\right] \geqslant \frac{1}{2}\lambda_{\min}(\Lambda_i)\|y^i\|^2 \tag{21}$$

and

$$H_i(y^i) = \frac{1}{2}(y^i)^T \Lambda_i y^i = \frac{1}{2}\left[\lambda_{i1}(y_i^1)^2 + \lambda_{i2}(y_i^2)^2\right] \leqslant \frac{1}{2}\lambda_{\max}(\Lambda_i)\|y^i\|^2, \tag{22}$$

where $i = 1,\ 2$.

It follows from (14), (18), (21), and (22) that

$$\frac{1}{2}\lambda_{\min}(\Lambda_i)\|y^i\|^2 \leqslant H_i(x) \leqslant \frac{1}{2}\lambda_{\max}(\Lambda_i)\|y^i\|^2. \tag{23}$$

Therefore, we know from (17), (20), and (23) that for any $x \in \mathbb{D} - \{x^{ei}\}$, the following holds.

$$\frac{1}{2}\lambda_{\min}(Q_i)d^2(x, x^{ei}) \leqslant H_i(x) \leqslant \frac{1}{2}\lambda_{\max}(Q_i)d^2(x, x^{ei}),\ i = 1,\ 2,$$

which is exact (12). The proof of Proposition 1 is thus completed. □

Since every subsystem of system (1) has a unique equilibrium point x^{ei}, $i = 1, 2$, and the maximum energy functions in Definition 1 are continuous everywhere in $\mathbb{D}$; there exists a unique compact region that is defined as

$$\Omega := \left\{z \in \mathbb{D} \,\middle|\, H_1(z) \leqslant g_1(x_1^{e2} - x_1^{e1})^2\right\} \bigcup \left\{z \in \mathbb{D} \,\middle|\, H_2(z) \leqslant g_2(x_1^{e1} - x_1^{e2})^2\right\} \subseteq \mathbb{D},\ \text{as}\ x_1^{e1} \neq x_1^{e2}. \tag{24}$$

or

$$\Psi := \left\{z \in \mathbb{D} \,\middle|\, H_1(z) \leqslant \eth_1(x_2^{e2} - x_2^{e1})^2\right\} \bigcup \left\{z \in \mathbb{D} \,\middle|\, H_2(z) \leqslant \eth_2(x_2^{e1} - x_2^{e2})^2\right\} \subseteq \mathbb{D},\ \text{as}\ x_2^{e1} \neq x_2^{e2}. \tag{25}$$

Similar to that of region stability defined in the reference[17], based on the region Ω in (24) or the region Ψ in (25), we introduce the concept of region stability for system (1) as follows.

Definition 3 [17] *Consider system (1) with the region Ω defined in (24) or Ψ defined in (25) under a special kind of switching path $\sigma(t)$, i.e., the switching line l_1 in (2). System (1) under the switching line l_1 in (2) is said to be*

- *Region stable with respect to the region Ω in (24) or the region Ψ in (25), if for $\forall\varepsilon > 0$, $\exists\delta := \delta(\varepsilon) > 0$ such that the following formula holds for any x_0,*

$$d(x_0, \Omega) < \delta \Longrightarrow d(x(t), \Omega) < \varepsilon,\ t \in [t_0, +\infty) \tag{26}$$

or

$$d(x_0, \Psi) < \delta \Longrightarrow d(x(t), \Psi) < \varepsilon,\ t \in [t_0, +\infty). \tag{27}$$

- *Asymptotically region stable with respect to the region Ω in (24) or the region Ψ in (25), if both (26) or (27) and the following limit hold*

$$\lim_{t\to+\infty} d(x(t), \Omega) = 0 \ \text{or}\ \lim_{t\to+\infty} d(x(t), \Psi) = 0. \tag{28}$$

3. REGION STABILITY ANALYSIS

This section will study the stability issue of switched two-dimensional linear Hamiltonian systems with ME. Based on the concept of region stability defined in Section 2, we propose several sufficient conditions of region stability and asymptotic region stability for system (1), respectively.

3.1. Some lemmas

This subsection introduces some lemmas that will be used in the next subsection. Firstly, it can be obtained from Proposition 1 that the following result holds.

Lemma 1 *Consider system (1) under the switching line (2). The Hamiltonian functions $H_1(x)$ and $H_2(x)$ and the maximum energy function $H(x)$ satisfy the following inequation.*

$$\frac{1}{2}\alpha d^2(x,\Omega) \leqslant H_i(x) \leqslant H(x) \leqslant \frac{1}{2}\beta d^2(x,\Omega),\ \forall x \in \mathbb{D}-\Omega,\ i=1,2, \tag{29}$$

where $\alpha = \min\{\lambda_{\min}(Q_1), \lambda_{\min}(Q_2)\}$, $\beta = \max\{\lambda_{\max}(Q_1), \lambda_{\max}(Q_2)\}$, and the region Ω and the maximum energy function $H(x)$ are defined in (24) and (10), respectively.

Proof: It is easy to see from the maximum energy function defined in (10) of Definition 1 and the equation (12) in Proposition 1 and Condition (24) that Lemma 1 holds true. □

Lemma 2 *Consider system (1) under the switching line (2). If $R_i \geqslant 0$, then for the trajectory $x(t) \in \mathbb{D}-\Omega$, $\forall t \in [t_m,\ t_{m+1})$, the two Hamiltonian functions $H_1(x)$ and $H_2(x)$ have the same variant trend of the properties of either monotonous increase or monotonous decrease at all the switching states x_m, where any $m \in \mathbb{N}$. That is, for all the switching states $x_m, x_{m+1} \in \mathbb{D}-\Omega$, $\triangle H_1 := H_1(x_m) - H_1(x_{m+1})$ and $\triangle H_2 := H_2(x_m) - H_2(x_{m+1})$ satisfy*

$$\mathrm{sign}(\triangle H_1) = \mathrm{sign}(\triangle H_2), \tag{30}$$

where $\mathrm{sign}(\cdot)$ denotes the sign function.

Proof: Since for $x(t) \in \mathbb{D}-\Omega$, $\forall t \in [t_m, t_{m+1})$, all the switching states x_m are situated in the switching line l_1 in (2), the following formula holds

$$H_{i_m}(x_m) = L_{i_m}(x_{1,m}) = g_{i_m}(x_{1,m} - x_1^{ei_m})^2, \tag{31}$$

where $x_{1,m}$ and $x_1^{ei_m}$ denote the first elements of the switching state x_m and the equilibrium point x^{ei_m}, respectively; And the parameters g_{i_m} are defined as follows.

$$g_{i_m} := \frac{1}{2}\begin{bmatrix}1\\k\end{bmatrix}^T Q_{i_m}\begin{bmatrix}1\\k\end{bmatrix} > 0,\ i_m \in \{1,2\},\ m \in \mathbb{N}.$$

It follows from the fact that the trajectory $x(t) \in \mathbb{D}-\Omega$ and $R_{i_m}(x) \geqslant 0$ that for $t \in [t_m, t_{m+1})$, $i_m = 1,2$, $m \in \mathbb{N}$, the Hamiltonian function $H_{i_m}(x)$ satisfies

$$\dot{H}_{i_m}(x) = -(x - x^{ei_m})^T Q_{i_m} R_{i_m} Q_{i_m}(x - x^{ei_m}) \leqslant 0. \tag{32}$$

One obtains from (32) that

$$H_{i_m}(x_{m+1}) \leqslant H_{i_m}(x_m),\ i_m = 1,2, \text{for any } m \in \mathbb{N}. \tag{33}$$

Since the first element $x_1^{ei_m}$ of the equilibrium point x^{ei_m} is contained in the region Ω, we know from (24), (29), (31), and (33) that for any $m \in \mathbb{N}$, $i_m = 1,2$,

$$L_{i_m}(x_{1,m+1}) \leqslant L_{i_m}(x_{1,m}) \Leftrightarrow d(x_{1,m}, x_1^{ei_m}) > d(x_{1,m+1}, x_1^{ei_m}) \Leftrightarrow d(x_m,\Omega) > d(x_{m+1},\Omega). \tag{34}$$

It is then obtained from (31) and (34) that for any $i_m \in \{1,\ 2\}$,

$$H_{i_m}(x_{m+1}) \leqslant H_{i_m}(x_m) \Leftrightarrow d(x_m,\Omega) > d(x_{m+1},\Omega). \tag{35}$$

One obtains from (31) and (35) that for all $i_m,\ i_s \in \Lambda$ and $i_m \neq i_s$, the following holds.

$$d(x_m, \Omega) > d(x_{m+1}, \Omega) \Longrightarrow L_{i_s}(x_{1,m+1}) \leqslant L_{i_s}(x_{1,m}) \Longrightarrow H_{i_s}(x_{m+1}) \leqslant H_{i_s}(x_m). \tag{36}$$

From (35) and (36), it follows that

$$\mathrm{sign}(\triangle H_1) = \mathrm{sign}(\triangle H_2), \tag{37}$$

which is (30). Thus, Lemma 2 holds true. □

Lemma 3 [27] *Let*

$$a_n^{\min} := \min_{i\in\Lambda}(a_n^i), \quad n = 1, 2, \ldots. \tag{38}$$

and

$$a_n^{\max} := \max_{i\in\Lambda}(a_n^i), \ n = 1, 2, \ldots. \tag{39}$$

If the two infinite sequences $\{a_n^i\}_{n=1}^{+\infty}$ *and* $i = 1, 2$ *are both monotonically decreasing/increasing, then* $\{a_n^{\min}\}_{n=1}^{+\infty}$ *and* $\{a_n^{\max}\}_{n=1}^{+\infty}$ *are also monotonically decreasing/increasing sequences.*

Lemma 4 [26] *System* (1) *under the switching line* l_1 *in* (2) *is region stable with respect to the region* Ω *in* (24) *if and only if for any* $x_0 \in \mathbb{D}$*, the following holds:*

$$H_{i_m}(x_m) \leqslant CH(x_0),\ i_m \in \Lambda,\ m \in \mathbb{N} \tag{40}$$

where the $i_m = \sigma(t_m) \in \{1, 2\}$*; The parameter* C *is a constant; And* $H(x)$ *is defined as in Definition 1.*

Remark 5 *Note that the proofs of Lemmas 1-4 are just related to the compact property of the regions of* Ω *in* (24) *and* Ψ *in* (25) *containing all the equilibrium points of the subsystems. Therefore, if the region* Ω *in* (24) *is replaced by the region* Ψ *in* (25)*, then all Lemmas 1-4 hold too.*

3.2. Regional stability results

Based on Definition 3, we obtain from Proposition 1, Lemmas 1-4 that the two main results of this paper are proposed in series as follows.

For the horizontal and vertical ordinates of the equilibrium points x^{e1} and x^{e2} of the two subsystems, there are the following two cases: (1) $x_1^{e1} \neq x_1^{e2}$; (2) $x_1^{e1} = x_1^{e2}$ and $x_2^{e1} \neq x_2^{e2}$. For the former, we propose the following region stability result of system (1) under the switching line l_1 in (2) as follows.

Theorem 1 *Consider system* (1) *with the compact region* Ω *in* (24) *and the switching line* l_1 *in* (2)*. For the case that* $x_1^{e1} \neq x_1^{e2}$*, system* (1) *under the switching line* l_1 *in* (2) *is*

(i) region stable with respect to the region Ω *in* (24)*, if* $J_1 \neq 0$, $J_2 \neq 0$, $R_1 \geqslant 0$, $R_2 \geqslant 0$*, and*

$$\frac{\beta}{\alpha} \geqslant \max\left\{1, 4\left(\frac{x_1^{e2} - x_1^{e1}}{v_1 - x_1^{e1}}\right)^2, 4\left(\frac{x_1^{e2} - x_1^{e1}}{v_1 - x_1^{e2}}\right)^2\right\}, \tag{41}$$

where $\alpha = \min\{\lambda_{\min}(Q_1), \lambda_{\min}(Q_2)\}$, $\beta = \max\{\lambda_{\max}(Q_1), \lambda_{\max}(Q_2)\}$*, and* v_1 *is the intersection point of the two parabolic curves* $L_1(x_1)$ *and* $L_2(x_1)$ *in* (6) *over the interval* (x_1^{e1}, x_1^{e2}) *in the horizontal axis, i.e.,*

$$v_1 = \begin{cases} \frac{(g_1 x_1^{e1} - g_2 x_1^{e2}) + \sqrt{g_1 g_2 (x_1^{e1} - x_1^{e2})^2}}{g_1 - g_2}, & \text{as } g_1 \neq g_2. \\ \frac{x_1^{e1} + x_1^{e2}}{2}, & \text{as } g_1 = g_2, \end{cases} \tag{42}$$

where g_1 *and* g_2 *are defined in* (7).

(ii) asymptotic region stable with respect to the region Ω*, if* $J_1 \neq 0$, $J_2 \neq 0$, $R_1 > 0$, $R_2 > 0$*, and the condition* (41) *are all satisfied.*

Proof: Without loss of generality, we just show that the two Statements (i) and (ii) hold for the case that $x_1^{e1} < x_1^{e2}$. As for the case that $x_1^{e1} > x_1^{e2}$, it is similar to show that the two Statements also hold true.

For any trajectory $x(t)$ of system (1) under the switching line l_1 in (2) starting from any initial state x_0 at the initial time t_0, there are the following three cases that should be considered: (a) The trajectory of the system $x(t) \in \mathbb{D} - \Omega, \forall t > t_0$; (b) The trajectory of the system $x(t), \forall t > t_0$, is always contained in Ω; (c) The trajectory of the system $x(t), \forall t > t_0$, is neither always contained in $\mathbb{D} - \Omega$ nor always contained in Ω.

(1) We show the conclusion of Theorem 1 holds for the case that $g_1 \neq g_2$.

Firstly, we will show that the Statement (i) holds for Case (a). Since the Hamiltonian functions $H_1(x)$ and $H_2(x)$ of the two subsystems are both continuous everywhere in $\mathbb{D}$, the maximum energy function $H(x)$ defined in Definition 1 is also continuous everywhere in $\mathbb{D}$. We obtain from the conditions of $Q_i > 0$, $R_i \geqslant 0$, where $i = 1, 2$, and the ordinary differential equations (ODEs) (1) that for any $t \in [t_m, t_{m+1})$ and $m \in \mathbb{N}$, the following holds.

$$\dot{H}_{i_m}(x) = -(x - x^{ei_m})^T Q_{i_m} R_{i_m} Q_{i_m} (x - x^{ei_m}) \leqslant 0. \tag{43}$$

It follows from (43) that

$$H_{i_m}(x(t_{m+1})) \leqslant H_{i_m}(x(t_m)), \ i_m = 1, 2, \quad \text{for any } m \in \mathbb{N}, \tag{44}$$

which can also be expressed as

$$H_{i_m}(x_{m+1}) \leqslant H_{i_m}(x_m), \ i_m = 1, 2, \quad \text{for any } m \in \mathbb{N}. \tag{45}$$

Since $R_i \geqslant 0$ holds for $i = 1, 2$, Lemma 2 holds for system (1). Then one knows from (30) in Lemma 2 and (45) that for $i_s \neq i_m \in \{1, 2\}$,

$$H_{i_s}(x_{m+1}) \leqslant H_{i_s}(x_m), \ \text{for any } m \in \mathbb{N}. \tag{46}$$

It can be obtained from (30) in Lemma 2, (45), and (46) that the two sequences $\{H_1(x_m)\}_{m=0}^{+\infty}$ and $\{H_2(x_m)\}_{m=0}^{+\infty}$ are both monotonically decreasing. It then can be obtained from Lemma 3 that the switching maximum energy sequence $\{H(x_m)\}_{m=0}^{+\infty}$ is also monotonically decreasing.

Based on the above analysis, one obtains from (10) in Definition 1 and the fact that the trajectory $x(t) \in \mathbb{D} - \Omega$, for all $t \in [t_m, t_{m+1})$, $i_m = 1, 2$, and any $m \in \mathbb{N}$ that

$$H_{i_m}(x(t_m)) \leqslant H(x(t_m)) \leqslant H(x(t_0)). \tag{47}$$

It is easy to see from (47) that

$$H_{i_m}(x(t_m)) \leqslant H(x(t_0)). \tag{48}$$

From which and (41), we know that

$$H_{i_m}(x_m) \leqslant \frac{\beta}{\alpha} H(x_0), \ i_m = 1, 2, \quad \text{for any } m \in \mathbb{N}. \tag{49}$$

It follows from (49) that (40) in Lemma 4 is satisfied for system (1). Then, by Lemma 4, we know that system (1) under the switching line l_1 in (2) is region stable with respect to the region Ω for Case (a).

Secondly, we will show that Statement (i) also holds for case (b). For g_1 and g_2 in (2), there are the following two relationships: $g_1 = g_2$ and $g_1 \neq g_2$. We first consider the case that $g_1 \neq g_2$. In this case, letting $L_1(x_1) = L_2(x_1)$ one obtains from (6) that

$$(g_1 - g_2)(x_1)^2 + (2g_2x_1^{e2} - 2g_1x_1^{e1})x_1 + g_1(x_1^{e1})^2 - g_2(x_1^{e2})^2 = 0. \tag{50}$$

From which, it can be known that the discriminant of the roots of the equation (50) is as follows.

$$\triangle = 4g_1g_2(x_1^{e1} - x_1^{e2})^2 > 0, \tag{51}$$

which means that two curves $L_1(x_1)$ and $L_2(x_1)$ have two intersection points.

It follows from (51) that the equation (50) has the following two solutions:

$$v_1 = \frac{\left(g_1x_1^{e1} - g_2x_1^{e2}\right) + \sqrt{g_1g_2\left(x_1^{e1} - x_1^{e2}\right)^2}}{g_1 - g_2} \tag{52}$$

and

$$v_2 = \frac{\left(g_1x_1^{e1} - g_2x_1^{e2}\right) - \sqrt{g_1g_2\left(x_1^{e1} - x_1^{e2}\right)^2}}{g_1 - g_2}, \tag{53}$$

where v_1 and v_2 denote the two intersection points of the two curves $L_1(x_1)$ and $L_2(x_1)$ satisfying $v_1 \in (x_1^{e1}, x_1^{e2})$ and $v_2 \in (-\infty, x_1^{e1}) \cup (x_1^{e2}, +\infty)$, respectively.

Next, we will find the intersection points of the switching line l_1 passing through the boundary of the region Ω. To do that, we obtain from $L_1(x_1) = L_1(x_1^{e2})$ that

$$(x_1)^2 - 2x_1^{e1}x_1 - (x_1^{e2})^2 + 2x_1^{e1}x_1^{e2} = 0. \tag{54}$$

It can be obtained from (54) that the discriminant of the roots of the equation (54) is as follows.

$$\triangle = 4(x_1^{e1} - x_1^{e2})^2 > 0. \tag{55}$$

It then follows from (55) that the solutions of the equation (54) are as follows.

$$x_1 = x_1^{e1} \pm (x_1^{e2} - x_1^{e1}). \tag{56}$$

The two intersection points can be denoted by $p_1 = x_1^{e2}$ and $p_2 = 2x_1^{e1} - x_1^{e2}$.

Similarly, solving $L_2(y_1) = L_2(x_1^{e1})$ yields that

$$y_1 = x_1^{e2} \pm (x_1^{e2} - x_1^{e1}) \tag{57}$$

The two intersections are then denoted by $p_3 = x_1^{e1}$ and $p_4 = 2x_1^{e2} - x_1^{e1}$, respectively.

One obtains from (56) and (57) that the switching line l_1 in (2) passes through the largest point and the smallest point of the boundary of the region Ω. The two maximum and minimum points are, respectively, denoted by

$$p_{\max} = \max_{i\in[4]}\{p_i\} = 2x_1^{e2} - x_1^{e1} \text{ and } p_{\min} = \min_{i\in[4]}\{p_i\} = 2x_1^{e1} - x_1^{e2}. \tag{58}$$

In the following, we consider the minimum value of the maximum energy function $H(x)$ over the region Ω.

(C1) As $v_2 \in (-\infty, x_1^{e1})$, the maximum energy function $H(x)$ is as follows.

$$H(x) = \begin{cases} L_2(x_1) = g_2(x_1 - x_1^{e2})^2, \text{ as } x_1 \in (v_2, v_1] \\ L_1(x_1) = g_1(x_1 - x_1^{e1})^2, \text{ as } x_1 \in [v_1, +\infty). \end{cases} \tag{59}$$

From (59), one obtains that the $L_2(x)$ is continuous over the interval $(v_2, v_1]$ and $\frac{dL_2(x_1)}{dx_1} < 0$, so the minimum value of $L_2(x_1)$ over the interval $(v_2, v_1]$ is $L_2(v_1)$.

Similarly, we know from (59) that the $L_1(x_1)$ is continuous over the interval $(v_1, +\infty)$ and $\frac{dL_1(x_1)}{dx_1} > 0$, so the minimum value of $L_1(x_1)$ over the interval $(v_1, +\infty)$ is $L_1(v_1)$.

It is obvious from (50) and (52) that

$$L_1(v_1) = L_2(v_1) \tag{60}$$

From (59) and (60), it shows that the minimum value of $H(x)$ under the case (C1) is as follows.

$$\inf \left\{ H(x) \,:\, x = [x_1 \; x_2]^T \in \mathbb{R}^2 \text{ and } x_1 \geqslant v_2 \right\} = \min \left\{ L_1(v_1),\, L_2(v_1) \right\} = L_1(v_1) = L_2(v_1), \tag{61}$$

where v_1 is expressed in (52).

(C2) As $v_2 \in \left(x_1^{e2}, +\infty\right)$, the maximum energy function $H(x)$ is as follows.

$$H(x) = \begin{cases} L_1(x_1) = g_1(x_1 - x_1^{e1})^2, \text{ as } x_1 \in [v_1,\, v_2] \\ L_2(x_1) = g_2(x_1 - x_1^{e2})^2, \text{ as } x_1 \in (-\infty, v_1]. \end{cases} \tag{62}$$

It can be obtained from (62) that the $L_2(x_1)$ is continuous over the interval $(-\infty, v_1)$ and $\frac{dL_2(x_1)}{dx_1} < 0$. Then, the minimum value of the maximum energy function $H(x) = L_2(x_1)$ on the interval $(-\infty, v_1]$ is $L_2(v_1)$.

Similarly, we know that the $L_1(x_1)$ is continuous over the interval $[v_1, v_2]$ and $\frac{dL_1(x_1)}{dx_1} > 0$. Then, the minimum value of the maximum energy function $H(x) = L_1(x_1)$ over the interval $[v_1, v_2]$ is $L_1(v_1)$.

Therefore, it can be obtained from (59) and (60) that the minimum value of the maximum energy function $H(x)$ under the case (C2) is as follows.

$$\inf \left\{ H(x) \,:\, x = [x_1 \; x_2]^T \in \mathbb{R}^2 \text{ and } x_1 \leqslant v_2 \right\} = \min \left\{ L_1(v_1), L_2(v_1) \right\} = L_1(v_1) = L_2(v_1). \tag{63}$$

Then, one obtains from (61) and (63) that the minimum value of the maximum energy function $H(x)$ over the region Ω is as follows.

$$\min \left\{ H(x) \,:\, x \in \Omega \right\} = L_1(v_1) = L_2(v_1). \tag{64}$$

It follows from (41) and the condition that $R_1 > 0$ and $R_2 > 0$ that

$$\frac{\beta}{\alpha} g_1(v_1 - x_1^{e1})^2 \geqslant g_1(p_{\max} - x_1^{e1})^2 \tag{65}$$

and

$$\frac{\beta}{\alpha} g_2(v_1 - x_1^{e2})^2 \geqslant g_2(p_{\min} - x_1^{e2})^2, \tag{66}$$

where $p_{\max}$ and $p_{\min}$ are the same as in (58).

From (59), (60), (62), (65), and (66), one obtains that

$$\frac{\beta}{\alpha}\min\left\{H(x)\ :\ x\in\Omega\right\}\geqslant\max\left\{L_1(p_{\max}),L_2(p_{\min})\right\},\tag{67}$$

where $\max\left\{L_1(p_{\max}),\ L_2(p_{\min})\right\}$ denotes the maximum value of $H(x)$ over the region Ω.

It is obvious from (64) and (67) that for any switching states x_m, the following holds.

$$\frac{\beta}{\alpha}H(x_0)\geqslant H_{i_m}(x_m),\ i_m=1,\ 2,\ \text{for any } m\in\mathbb{N},\tag{68}$$

which implies that (40) in Lemma 4 is satisfied. By Lemma 4, we know that for Case (b), system (1) under the switching line l_1 in (2) is region stable with respect to the region Ω.

Thirdly, we show that Statement (i) also holds for Case (c) as follows. In such case, the trajectory $x(t)$ of system (1) is not always in the region $\mathbb{D}-\Omega$. For any small enough $\varepsilon>0$, there is a time interval sequence $\left\{\left[t_{m_\tau},t_{m_\tau+1}\right]\right\}_{\tau=1}^{+\infty}$, such that the states $x_{m_\tau}=x(t_{m_\tau};t_0,x_0,i_{m_\tau})$ and $x_{m_\tau+1}=x(t_{m_\tau+1};t_0,x_0,i_{m_\tau+1})$ are all on the boundary of the set $\Omega_\varepsilon:=\left\{z\in\mathbb{D}\colon \mathrm{d}(z,\Omega)\leq\varepsilon\right\}$. We insert x_{m_τ} and $x_{m_\tau+1}$ into the switching state sequence $\left\{x_m\right\}_{m=0}^{+\infty}$. Correspondingly, the times t_{m_τ} and $t_{m_\tau+1}$ are inserted into the switching time sequence $\left\{t_m\right\}_{\mathrm{m=0}}^{+\infty}$, and the indexes i_{m_τ} and $i_{m_\tau+1}$ are all inserted into the switching index sequence $\left\{i_m\right\}_{\mathrm{m=0}}^{+\infty}$.

Based on the above statements, we know that there is a time interval $\left[t_{m_\tau},t_{m_\tau+1}\right]$ for $m_\tau\in\mathbb{N}$, such that for $\tau\in\mathbb{N}$, $i_{m_\tau},i_{m_\tau+1}=1,\ 2,$

$$x(t\colon t_{m_\tau},x_{m_\tau},i_{m_\tau})\in\mathbb{D}-int(\Omega_\varepsilon)\subset\mathbb{D}-\Omega\tag{69}$$

It is known from (69) that Statement (i) is true for the trajectory of $x(t\colon t_{m_\tau},x_{m_\tau},i_{m_\tau})$ of system (1) contained in $\mathbb{D}-\mathrm{int}(\Omega_\varepsilon)$. Thus the proof is similar to that proof of Case (a). On the other hand, it is obvious that as $\varepsilon\to 0$, The state trajectory $x(t)$ contained in the region $\Omega_\varepsilon-\Omega$ will gradually go into the region Ω. Based on the above, we know that Statement (i) holds too.

Finally, Statement (ii) will be shown under the following two Situations.

(S1) As $v_2\in(-\infty,x_1^{e1})$ and $x(t)\in\mathbb{D}-\Omega$, for all $t\in[t_m,t_{m+1})$ and any $m\in\mathbb{N}$, the values of the maximum energy function $H(x)$ at the switching states $x_m=[x_{1,m}\ x_{2,m}]^T\in\mathbb{R}^2$ and $m\in\mathbb{N}$ are as follows.

$$H(x_m)=\begin{cases}L_2(x_{1,m})=g_2(x_{1,m}-x_1^{e2})^2, & x_{1,m}\in(v_2,v_1]\\ L_1(x_{1,m})=g_1(x_{1,m}-x_1^{e1})^2, & x_{1,m}\in(-\infty,v_2)\cup(v_1,+\infty)\end{cases}\tag{70}$$

The subsystem i_m is activating when any $t\in[t_m,t_{m+1})$ and $x(t)\in\mathbb{D}-\Omega$. And it is obvious from Condition $R_1>0$ and $R_2>0$ of Statement (ii) that for any $i_m=1,\ 2,$

$$\dot{H}_{i_m}=-(x_m-x^{ei_m})^TQ_{i_m}R_{i_m}Q_{i_m}(x_m-x^{ei_m})<0,\ i_m=1,\ 2,\ m\in\mathbb{N}.\tag{71}$$

We know from (71) that

$$H_{i_m}(x(t_{m+1}))<H_{i_m}(x(t))<H_{i_m}(x(t_m)),\ i_m=1,\ 2,\ m\in\mathbb{N}.\tag{72}$$

From Lemma 2, the condition of $R_1>0$ and $R_2>0$, (38), (39), (41), and (43), one can show that the two sequences of $\left\{L_1(x_{1,m})\right\}_m^{+\infty}$ and $\left\{L_2(x_{1,m})\right\}_m^{+\infty}$ both monotonically decrease over the interval $\left(-\infty,x_1^{e1}\right)\cup(x_1^{e2},+\infty)$. Then, the switching maximum energy sequence of $\{H(x_m)\}_m^{+\infty}$ also decreases monotonically over the following time interval: $\left(-\infty,\ x_1^{e1}\right)$.

On the other hand, one obtains from Lemma 2 and (72) that

$$H(x(t_{m+1})) < H(x(t)) < H(x(t_m)),\ i_m = 1,\ 2,\ m \in \mathbb{N}. \tag{73}$$

Then, as $m \to \infty$ and all the horizontal ordinates $x_{1,m}$ of switching states x_m are contained in the interval $(-\infty, p_{\min}) \subset (-\infty, x_1^{e1})$, where $p_{\min}$ and x_1^{e1} satisfy $p_{\min} \leqslant x_1^{e1} \leqslant v_1 \leqslant x_1^{e2}$. And it follows from (73) that the following holds true.

$$\lim_{m\to\infty} H(x(t_m)) = \lim_{m\to\infty} H(x(t_{m+1})) = L_2(p_{\min}) = g_2(p_{\min} - x_1^{e2})^2, \tag{74}$$

where $p_{\min}$ and g_2 are the same as in (58) and (7), respectively.

By the squeeze theorem, one obtains from (73) and (74) that

$$\lim_{t\to\infty} H(x(t)) = L_2(p_{\min}) = g_2(p_{\min} - x_1^{e2})^2, \tag{75}$$

where $p_{\min}$ and g_2 are the same as in (58) and (7), respectively.

As $m \to \infty$ and all the horizontal ordinates $x_{1,m}$ of switching states x_m are contained in the interval as follows. $(p_{\max}, +\infty) \subset (x_1^{e2}, +\infty)$, where $p_{\min} \leqslant x_1^{e1} \leqslant v_1 \leqslant x_1^{e2} \leq p_{\max}$. One knows that

$$\lim_{m\to\infty} H(x(t_m)) = \lim_{m\to\infty} H(x(t_{m+1}) = L_1(p_{\max}) = g_1(p_{\max} - x_1^{e1})^2. \tag{76}$$

It is obvious from (72) and (76) and the squeeze theorem that

$$\lim_{m\to\infty} H(x(t)) = L_1(p_{\max}) = g_1(p_{\max} - x_1^{e1})^2, \tag{77}$$

where $p_{\max}$ and g_1 are the same as in (58) and (7), respectively.

Since $L_1(p_{\max})$ and $L_2(p_{\min})$ are contained in the region Ω, it can be seen from (24), (73), (74), and (76) that

$$\lim_{t\to+\infty} d(x(t), \Omega) = 0. \tag{78}$$

(S2) As $v_2 \in \left(x_1^{e2}, +\infty\right)$ and the trajectory $x(t) \in \mathbb{D} - \Omega$, for any $t \in [t_m, t_{m+1})$, $m \in \mathbb{N}$, the switching maximum energy sequence $H(x_m)$ corresponding to switching states $x_m = [x_{1,m}\ x_{2,m}]^T \in \mathbb{R}^2$ and $m \in \mathbb{N}$ is as follows.

$$H(x_m) = \begin{cases} L_2(x_{1,m}) = g_2(x_{1,m} - x_1^{e2})^2, & \text{as } x_{1,m} \in (-\infty, v_1) \cup (v_2, +\infty), \\ L_1(x_{1,m}) = g_1(x_{1,m} - x_1^{e1})^2, & \text{as } x_{1,m} \in [v_1, v_2], \text{ for all } m \in \mathbb{N}, \end{cases} \tag{79}$$

where the two functions $L_1(\cdot)$ and $L_2(\cdot)$ are defined in (6). g_1 and g_2 are in (7).

Similar to that proof of Situation (S1), one obtains from Lemma 2 and the condition of $R_1 > 0$ and $R_2 > 0$, (6), (38), (39), (73), and (43) that (78) also holds for this Situation. It thus follows from the above considerations of Situations (S1) and (S2) that Statement (*ii*) holds for Case (a).

As for Case (b), it is easy to see from the conditions of Statement (*ii*) that system (1) under the switching line l_1 in (2) is region stable with respect to the region Ω defined in (6). Meanwhile, (28) of Definition 3 follows from the fact that the system trajectory $x(t)$ is always contained in the region Ω. Thus, all the conditions of Definition (3) are satisfied for system (1). Therefore, the correctness of Statement (*ii*) follows from Definition (3) for Case (b).

Finally, we show that Statement (ii) also holds for Case (c) as follows. Similar to that proof of the two cases (C1) and (C2) of Situation (i), one can show from (69) that Statement (ii) holds true for the case that the whole trajectory $x(t\colon t_{m_l}, x_{m_l}, i_{m_l})$ of system (1) is contained in $\mathbb{D} - \mathrm{int}(\Omega_\varepsilon)$. On the other hand, it is obvious that as $\varepsilon \to 0$, The state trajectory $x(t)$ contained in the region $\Omega_\varepsilon - \Omega$ will gradually converge into the region Ω.

Based on the above, by Definition 3, we obtain from the condition of $R_1 > 0$ and $R_2 > 0$, (32), (74), (76), and (78) that system (1) under the switching line l_1 in (2) is asymptotically region stable with respect to the region Ω. That is, Statement (ii) holds.

(2) We show the conclusion of Theorem 1 also holds for the case that $g_1 = g_2$.

In this case, letting $L_1(x_1) = L_2(x_1)$, together with (6), yields

$$2(x_1^{e2} - x_1^{e1})x_1 + (x_1^{e1})^2 - (x_1^{e2})^2 = 0. \tag{80}$$

Solving the above equation (80), we obtain its solution denoted by v_1 as follows.

$$v_1 = \frac{x_1^{e1} + x_1^{e2}}{2}. \tag{81}$$

It can be obtained from (81) that $p_{\max} = 2x_1^{e2} - x_1^{e1}$ and $p_{\min} = 2x_1^{e1} - x_1^{e2}$ being the two points on the boundary of the region Ω. Then, similar to that of the case of $g_1 \neq g_2$, it can be shown that Theorem 1 also holds for the case that $g_1 = g_2$.

Thus, the proof of Theorem 1 is finished. □

For the case that horizontal ordinates of the equilibrium points x^{e1} and x^{e2} of the two subsystems are the same but their vertical ordinates are different, i.e., $x_1^{e1} = x_1^{e2}$ and $x_2^{e1} \neq x_2^{e2}$, we present another main result of this paper as follows.

Theorem 2 *Consider system* (1) *with the compact region* Ψ *in* (25) *and the switching line* l_1 *in* (2). *For the case that* $x_2^{e1} \neq x_2^{e2}$, *system* (1) *under the switching line* l_1 *in* (2) *is*

(i) *region stable with respect to the region* Ψ *in* (25), *if* $J_1 \neq 0$, $J_2 \neq 0$, $R_1 \geqslant 0$, $R_2 \geqslant 0$, *and*

$$\frac{\beta}{\alpha} \geqslant \max\left\{1, 4\left(\frac{x_2^{e2} - x_2^{e1}}{v_2 - x_2^{e1}}\right)^2, 4\left(\frac{x_2^{e2} - x_2^{e1}}{v_2 - x_2^{e2}}\right)^2\right\}, \tag{82}$$

where $\alpha = \min\left\{\lambda_{\min}(Q_1), \lambda_{\min}(Q_2)\right\}$, $\beta = \max\left\{\lambda_{\max}(Q_1), \lambda_{\max}(Q_2)\right\}$, *and* v_2 *is the intersection point of the two parabolic curves* $M_1(x_2)$ *and* $M_2(x_2)$ *in* (8) *over the interval* (x_2^{e1}, x_2^{e2}) *in the vertical axis, i.e.,*

$$v_2 = \begin{cases} \frac{(\eth_1 x_2^{e1} - \eth_2 x_2^{e2}) + \sqrt{\eth_1 \eth_2 \left(x_2^{e1} - x_2^{e2}\right)^2}}{\eth_1 - \eth_2}, & \text{as } \eth_1 \neq \eth_2. \\ \frac{x_2^{e1} + x_2^{e2}}{2}, & \text{as } \eth_1 = \eth_2. \end{cases} \tag{83}$$

where $\eth_1$ *and* $\eth_2$ *are defined in* (9).

(ii) *asymptotic region stable with respect to the region* Ω, *if* $J_1 \neq 0$, $J_2 \neq 0$, $R_1 > 0$, $R_2 > 0$, *and the condition* (82) *are all satisfied.*

Proof: Similar to that proof of Theorem 1, we show from $x_2^{e1} \neq x_2^{e2}$ that Theorem 2 also holds via replacing all the horizontal ordinates x_1^{e1} and x_1^{e2} of the equilibrium points x^{e1} and x^{e2} of the two subsystems by their vertical ordinates of x_2^{e1} and x_2^{e2} during the whole proof of Theorem 1. □

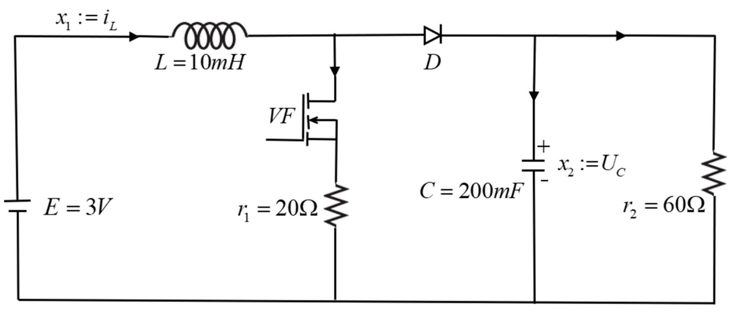

Figure 2. Switching circuit schematic.

4. AN APPLICATION AND TWO NUMERICAL EXAMPLES

In this section, the main results obtained in Section 3 are applied to a switching electric circuit in Subsection 4.1. Two numerical examples are carried out to verify the effectiveness and practicability of Theorems 1 and 2, respectively, proposed in Section 3.

4.1. An application to a switching DC electric circuit

Consider an ideal switching DC electric circuit, as shown in Figure 2. In which L, C, r_1, r_2, E, and VF denote the inductor, the capacitor, the two resistors, the DC electric source, and the switch, respectively. For simplicity, the notations of L, C, r_1, r_2, and E also denote the inductance, the capacitance, the resistances, and the electric voltage, respectively, of the corresponding circuit elements. The constant parameters of these circuit elements are chosen as follows. $L = 10mH$, $C = 200mF$, $r_1 = 20\Omega$, $r_2 = 60\Omega$, and $E = 3V$. For the switch VF, there are always the following two actions: the OFF and the ON. Therefore, such an electric circuit is essential to a switching DC electric circuit. By the famous Kirchhoff's voltage law and Kirchhoff's current law, we can model the electric circuit by a switched linear Hamiltonian system, such as system (1).

To do that, we let $x = [x_1, x_2]^T$ be the state of the switching DC electric circuit situated in Figure 2. Where the first element x_1 of the state x denotes the electric current $i_L(t)$ of the inductor L, and the second element x_2 of the state x denotes the voltage $U_C(t)$ of the capacitor C. Then, the electric circuit is modeled as a switched dissipative linear Hamiltonian system with two subsystems as follows. The following two subsystems are expressed as

$$\begin{cases} \dot{x} = [J_i - R_i]\nabla H_i(x - x^{ei}), \\ x(0) = x_0, \end{cases} \quad \text{for all } t \geqslant 0,\ i = 1,\ 2, \tag{84}$$

where the corresponding matrices of the first and second subsystems are as follows.

$$J_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},\ R_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0.00083 \end{bmatrix},\ J_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},\ \text{and } R_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0.0033 \end{bmatrix}. \tag{85}$$

The Hamiltonian functions of the two subsystems are, respectively, listed as follows.

$$H_i(x) = 0.5(x - x^{ei})Q_i(x - x^{ei})^T\ i = 1,\ 2, \tag{86}$$

in which the matrices Q_1 and Q_2 are listed in the following:

$$Q_1 = \begin{bmatrix} 5 & 0 \\ 0 & 100 \end{bmatrix} \text{ and } Q_2 = \begin{bmatrix} 5 & 0 \\ 0 & 100 \end{bmatrix}. \tag{87}$$

The equilibrium points of the first subsystem and the second subsystem are $x^{e1} = [0.05\ 3]^T$ and $x^{e2} = [0.2\ 3]^T$, respectively. The equilibrium points x^{e1} and x^{e2} of the two different subsystems are simultaneously passed through a switching line l_1 expressed as

$$l_1 \,:\, x_2 = 3, \tag{88}$$

which is a special switching path $\sigma(t)$ governing the activation of the two subsystems in system (84).

One obtains from (85) and (87) that

$$[(J_1 - R_1)Q_1][(J_2 - R_2)Q_2] = \begin{bmatrix} -500 & 33.33 \\ -0.42 & -499.97 \end{bmatrix} \tag{89}$$

and

$$[(J_2 - R_2)Q_2][(J_1 - R_1)Q_1] = \begin{bmatrix} -500 & 8.33 \\ -1.67 & -499.97 \end{bmatrix}. \tag{90}$$

From (89) and (90), it can be seen that although the two subsystems of system (84) are two essential linear systems, their state matrices $(J_1 - R_1)Q_1$ and $(J_2 - R_2)Q_2$ do not commute each other. Therefore, the region stability of system (84) cannot be verified by the stability criteria given in the reference[17]. Moreover, there are not any other stability criteria reported in the open literature that can be used to check the stability of system (84). However, by the main results of this paper, we can check the stability of system (84) as follows.

It can be obtained from (87), (7), (42), and the equilibrium points $x^{e1} = [0.05\ 3]^T$ and $x^{e2} = [0.2\ 3]^T$ of the two subsystems that

$$\alpha = \min\left\{\lambda(Q_1),\, \lambda(Q_2)\right\} = 5,\ \beta = \max\left\{\lambda(Q_1),\, \lambda(Q_2)\right\} = 100 \tag{91}$$

and

$$v_1 = \max\left\{1, 4\left(\frac{x_2^{e2} - x_2^{e1}}{v_2 - x_2^{e1}}\right)^2, 4\left(\frac{x_2^{e2} - x_2^{e1}}{v_2 - x_2^{e2}}\right)^2\right\} = 16. \tag{92}$$

It is obvious from (91) and (92) that (41) of Theorem 1 is satisfied. One knows from (85) that the following hold: $J_1 \neq 0$, $J_2 \neq 0$, $R_1 \geqslant 0$, and $R_2 \geqslant 0$, which are also satisfied for system (84) under the switching line l_1 in (88). Therefore, all the conditions of Conclusion (i) of Theorem 1 are satisfied for system (84). By Theorem 1, we know that system (84) under the switching line l_1 in (88) is region stable with respect to the following region expressed as

$$\begin{aligned} \Omega &= \left\{z \in \mathbb{R}^2 \,:\, H_1(z) \leqslant \max_{i=1,2}\left\{H_i(x^{e2})\right\}\right\} \bigcup \left\{z \in \mathbb{R}^2 \,:\, H_2(z) \leqslant \max_{i=1,2}\left\{H_i(x^{e1})\right\}\right\} \\ &= \left\{z \in \mathbb{R}^2 \,:\, H_1(z) \leqslant 0.0563\right\} \bigcup \left\{z \in \mathbb{R}^2 \,:\, H_2(z) \leqslant 0.0563\right\}. \end{aligned} \tag{93}$$

To show the above conclusion, we will simulate system (84) as follows. An initial state is chosen as $x_0 = [0.03\ 2]^T$, which is contained in the exterior of the region Ω in (93). The simulations are carried out and illustrated in Figures 3-5. Figure 3 denotes the response of switching path $\sigma(t)$ in relation to the switching line l_1 in (88) with respect to time t. Figure 4 denotes the trajectory of system (84) under the switching path σ in relation to the switching line l_1 starting from the initial state x_0 with respect to time t. Figure 5 denotes the trajectory of system (84) under the switching path σ in relation to the switching line l_1 in the plane $\mathbb{R}^2$ starting from the initial state x_0. It is easy to see from Figure 5 that the trajectory $x(t)$ of system (84) goes to or into the region Ω as $t \to +\infty$. Therefore, the simulations show the effectiveness and practicality of Conclusion (i) of Theorem 1.

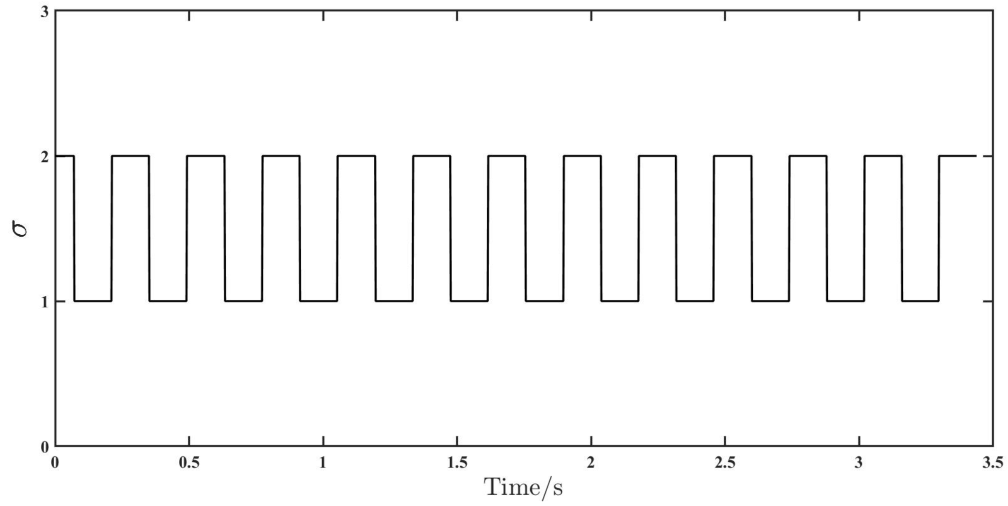

Figure 3. The response of the switching path σ in relation to the switching line l_1 with respect to time t.

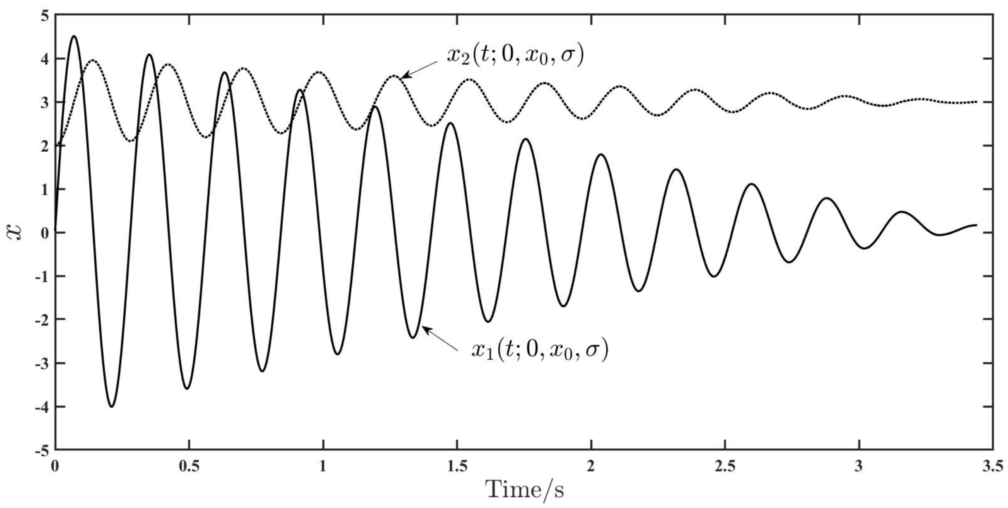

Figure 4. The trajectory of system (84) under the switching path σ in relation to the switching line l_1 starting from the initial state x_0 with respect to time t.

4.2. Two numerical examples

Example 1 *Consider a switched linear Hamiltonian system with two subsystems as follows.*

$$\begin{cases} \dot{x} = [J_i - R_i]\nabla H_i(x), \\ x(0) = x_0, \end{cases} \quad \text{for all } t \geqslant 0,\ i = 1,\ 2, \tag{94}$$

governed by a switching path $\sigma(t) : [0, +\infty) \to \{1, 2\}$, *which is also a switching line passing through the two points,* $x^{e1} = [1\ 2]^T$ *and* $x^{e2} = [8\ 16]^T$, *of the equilibrium points of the first subsystem and the second subsystem, respectively. That is, the switching line can be expressed as*

$$l_1 : x_2 = 2x_1. \tag{95}$$

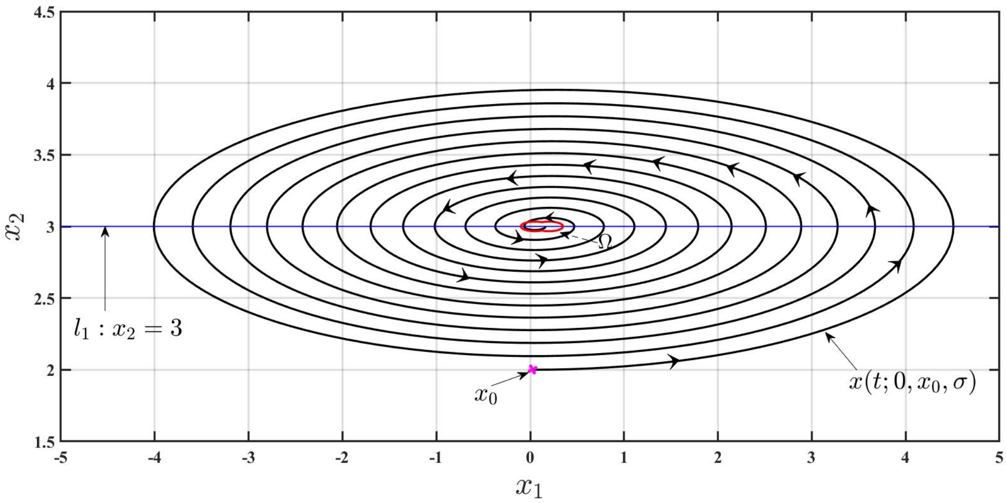

Figure 5. The trajectory of system (84) under the switching path σ in relation to the switching line l_1 in the plane $\mathbb{R}^2$ starting from the initial state x_0.

In system (94), the Hamiltonian functions of the two subsystems are listed as

$$H_i(x) = 0.5\left(x - x^{ei}\right)Q_i\left(x - x^{ei}\right)^T \; i = 1, 2, \tag{96}$$

where the matrices Q_1 and Q_2 are, respectively, listed as the corresponding matrices of the subsystems as follows.

$$Q_1 = \begin{bmatrix} 18 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } Q_2 = \begin{bmatrix} 15 & 0 \\ 0 & 1 \end{bmatrix}. \tag{97}$$

The corresponding matrices of the first and second subsystems are as follows.

$$J_1 = \begin{bmatrix} 0 & -8 \\ 8 & 0 \end{bmatrix}, \; R_1 = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}, \; J_2 = \begin{bmatrix} 0 & -12 \\ 12 & 0 \end{bmatrix}, \text{ and } R_2 = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}. \tag{98}$$

We obtain from (97) and (98) that

$$[(J_1 - R_1)Q_1][(J_2 - R_2)Q_2] = \begin{bmatrix} 1800 & 656 \\ -9000 & -1726 \end{bmatrix} \tag{99}$$

and

$$[(J_2 - R_2)Q_2][(J_1 - R_1)Q_1] = \begin{bmatrix} 1512 & 504 \\ -9864 & -1438 \end{bmatrix}. \tag{100}$$

From (99) and (100), it is obvious that although the two subsystems of system (94) are two essential linear systems, their state matrices $(J_1 - R_1)Q_1$ and $(J_2 - R_2)Q_2$ cannot commute. Therefore, the region stability of system (94) cannot be verified by the stability criteria obtained in the reference[17]. Moreover, there is not any stability criteria reported in the open literature. However, we can check the stability of system (94) as follows.

According to (7), (97), and (41), we obtain that

$$\alpha = \min\left\{\lambda(Q_1), \, \lambda(Q_2)\right\} = 1, \; \beta = \max\left\{\lambda(Q_1), \, \lambda(Q_2)\right\} = 18 \tag{101}$$

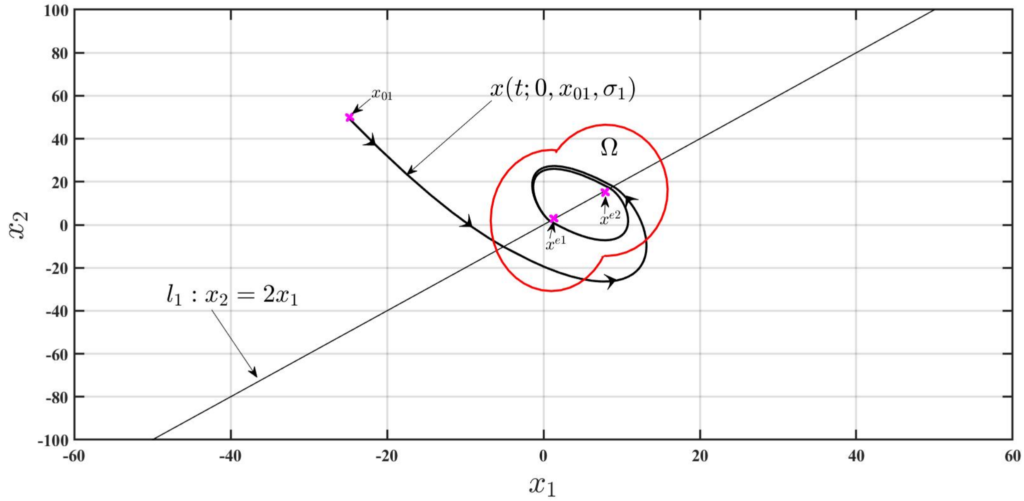

Figure 6. The trajectory of system (94) under the switching path σ_1 in relation to the switching line l_1 in the plane $\mathbb{R}^2$ starting from the initial state x_{01}.

and

$$v_1 = \max\left\{1, 4\left(\frac{x_2^{e2}-x_2^{e1}}{v_2-x_2^{e1}}\right)^2, 4\left(\frac{x_2^{e2}-x_2^{e1}}{v_2-x_2^{e2}}\right)^2\right\} = 4.3718. \tag{102}$$

It is obvious from (101) and (102) that (41) of Theorem 1 is satisfied.

It is obvious from (98), (101), and (102) that $J_1 \neq 0$, $J_2 \neq 0$, $R_1 > 0$, $R_2 > 0$, and (41) are all satisfied for system (94) under the switching line l_1 denoted in (95). That is, all the conditions of Conclusion (*ii*) of Theorem 1 are satisfied for system (94). By Theorem 1, we know that system (94) under the switching line l_1 in (95) is asymptotically region stable with respect to the following region:

$$\begin{aligned}\Omega &= \left\{z \in \mathbb{R}^2 : H_1(z) \leqslant \max_{i=1,2}\left\{H_i(x^{e2})\right\}\right\} \bigcup \left\{z \in \mathbb{R}^2 : H_2(z) \leqslant \max_{i=1,2}\left\{H_i(x^{e1})\right\}\right\} \\ &= \left\{z \in \mathbb{R}^2 : H_1(z) \leqslant 539\right\} \bigcup \left\{z \in \mathbb{R}^2 : H_2(z) \leqslant 465.5\right\}. \end{aligned} \tag{103}$$

To show the above conclusion by simulations, we choose the following two initial states: $x_{01} = [-25\ 50]^T$ and $x_{02} = [4\ 8]^T$. It is easy to see that the two initial states x_{01} and x_{02} are contained in the interior and the exterior of the region Ω in (103), respectively. The numerical simulations are then carried out, and the results are shown in Figures 6-11, which denote the trajectories of system (94) under the switching line l_1 in (95) starting from the two initial states in the plane $\mathbb{R}^2$, the trajectories with respect to time t, and switching path with respect to time t, respectively. It can be seen from Figures 6 and 7 that the trajectory $x(t)$ of system (94) goes to/into the region Ω as $t \to +\infty$. Therefore, these simulations show that the Conclusion (*ii*) of Theorem 1 is effective and practical.

Example 2 *Consider a switched linear Hamiltonian system with two subsystems as follows.*

$$\begin{cases} \dot{x} = [J_i - R_i]\nabla H_i(x), \\ x(0) = x_0, \end{cases} \quad \text{for all } t \geqslant 0,\ i = 1,\ 2, \tag{104}$$

governed by a switching path $\sigma(t) : [0, +\infty) \to \{1, 2\}$, *which is also a switching line passing through the two points,* $x^{e1} = [2\ 2]^T$ *and* $x^{e2} = [2\ 12]^T$, *of the equilibrium points of the first subsystem and the second subsystem,*

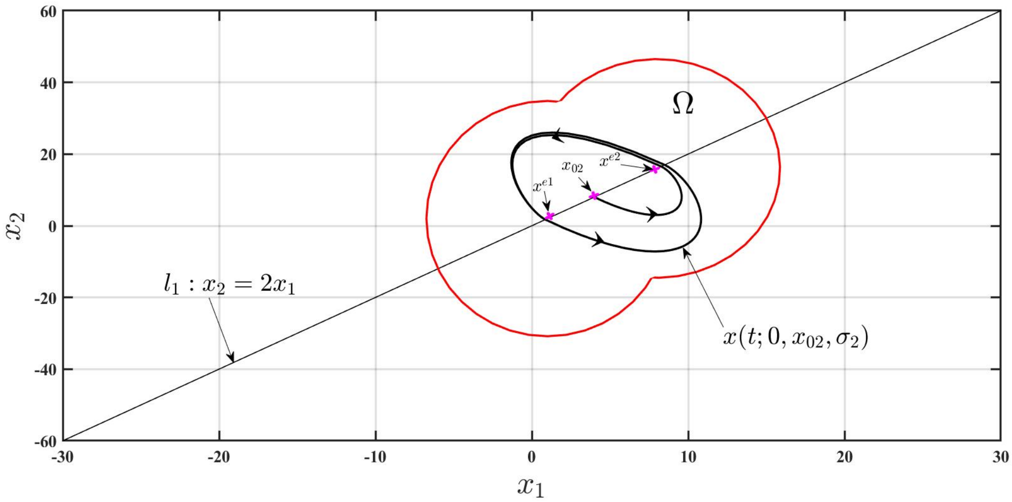

Figure 7. The trajectory of system (94) under the switching path σ_2 in relation to the switching line l_1 in the plane $\mathbb{R}^2$ starting from the initial state x_{02}.

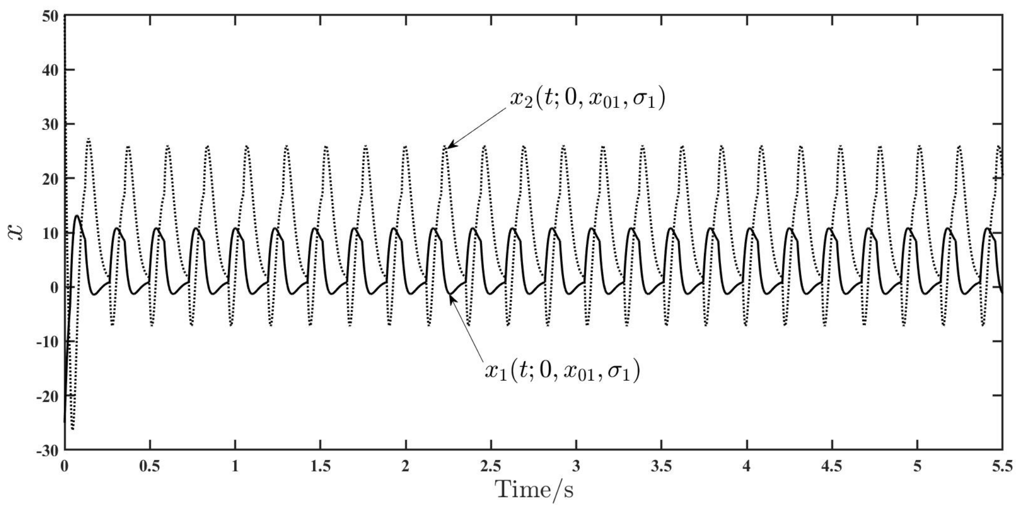

Figure 8. The trajectory of system (94) under the switching path σ_1 in relation to the switching line l_1 starting from the initial state x_{01} with respect to time t.

respectively. That is, the switching line can be expressed as

$$l_1 \ : \ x_1 = 2. \tag{105}$$

In system (104), the Hamiltonian functions of the two subsystems are listed as

$$H_i(x) = 0.5(x - x^{ei})Q_i(x - x^{ei})^T \ i = 1, 2, \tag{106}$$

where the matrices Q_1 and Q_2 are, respectively, listed as the corresponding matrices of the subsystems as follows.

$$Q_1 = \begin{bmatrix} 1 & 0 \\ 0 & 15 \end{bmatrix} \text{ and } Q_2 = \begin{bmatrix} 15 & 0 \\ 0 & 18 \end{bmatrix}. \tag{107}$$

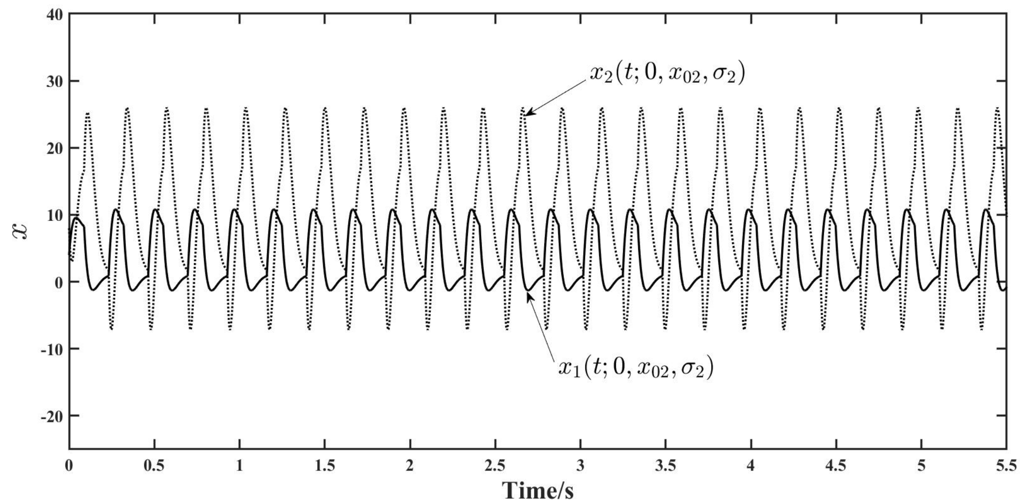

Figure 9. The trajectory of system (94) under the switching path σ_2 in relation to the switching line l_1 starting from the initial state x_{02} with respect to time t.

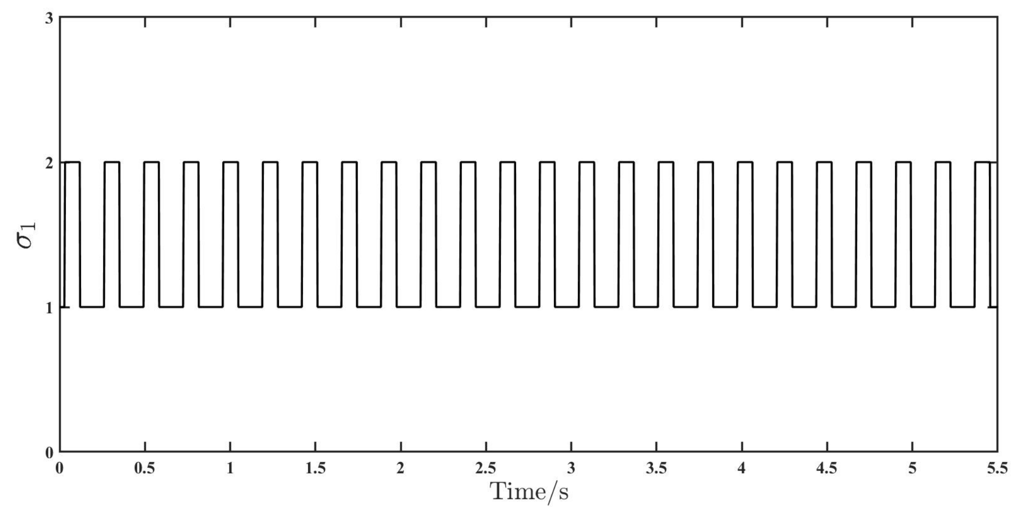

Figure 10. The response of the switching path σ_1 in relation to the switching line l_1 with respect to time t.

The corresponding matrices of the first and second subsystems are as follows.

$$J_1 = \begin{bmatrix} 0 & -10 \\ 10 & 0 \end{bmatrix}, \ R_1 = \begin{bmatrix} 4 & 0 \\ 0 & 5 \end{bmatrix}, \ J_2 = \begin{bmatrix} 0 & -8 \\ 8 & 0 \end{bmatrix}, \text{ and } R_2 = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}. \tag{108}$$

One obtains from (104), (107), and (108) that the following two:

$$[(J_1 - R_1)Q_1][(J_2 - R_2)Q_2] = \begin{bmatrix} -17820 & 5976 \\ -9450 & 1260 \end{bmatrix} \tag{109}$$

and

$$[(J_2 - R_2)Q_2][(J_1 - R_1)Q_1] = \begin{bmatrix} -1260 & 17550 \\ -840 & -15300 \end{bmatrix}. \tag{110}$$

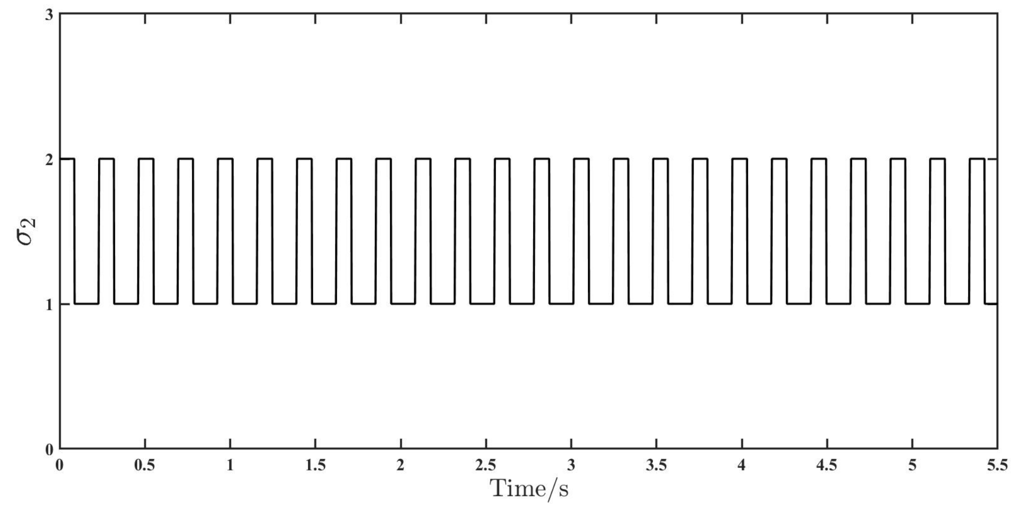

Figure 11. The response of the switching path σ_2 in relation to the switching line l_1 with respect to time t.

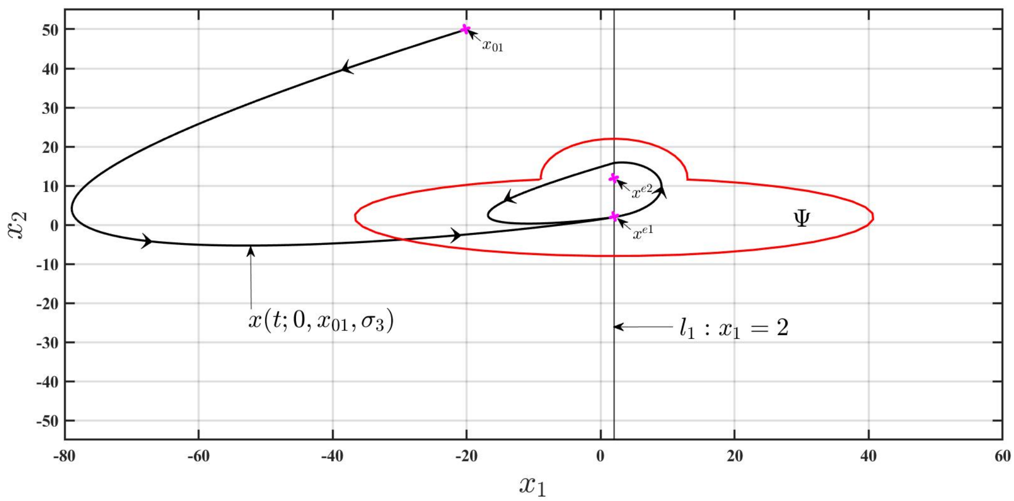

Figure 12. The trajectory of system (104) under the switching path σ_3 in relation to the switching line l_1 in the plane $\mathbb{R}^2$ starting from the initial state x_{01}.

From (109) and (110), it is obvious that although the two subsystems of system (104) are also two linear systems, their state matrices $(J_1 - R_1)Q_1$ and $(J_2 - R_2)Q_2$ are not commutative. Therefore, the region stability of system (104) cannot be verified by the stability criteria obtained in the reference[17]. Moreover, there is not any stability criteria reported in the open literature. However, we can check the stability of system (104) as follows.

According to (7), (107), and (41), we obtain that

$$\alpha = \min\left\{\lambda(Q_1),\ \lambda(Q_2)\right\} = 1,\ \beta = \max\left\{\lambda(Q_1),\ \lambda(Q_2)\right\} = 18, \tag{111}$$

and

$$v_2 = \max\left\{1, 4\left(\frac{x_2^{e2} - x_2^{e1}}{v_2 - x_2^{e1}}\right)^2, 4\left(\frac{x_2^{e2} - x_2^{e1}}{v_2 - x_2^{e2}}\right)^2\right\} = 7.2277. \tag{112}$$

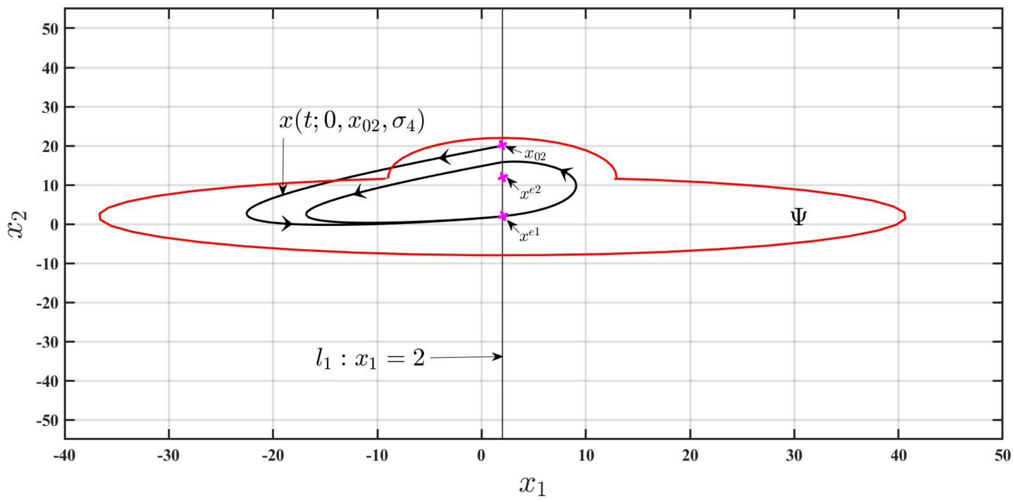

Figure 13. The trajectory of system (104) under the switching path σ_4 in relation to the switching line l_1 in the plane $\mathbb{R}^2$ starting from the initial state x_{02}.

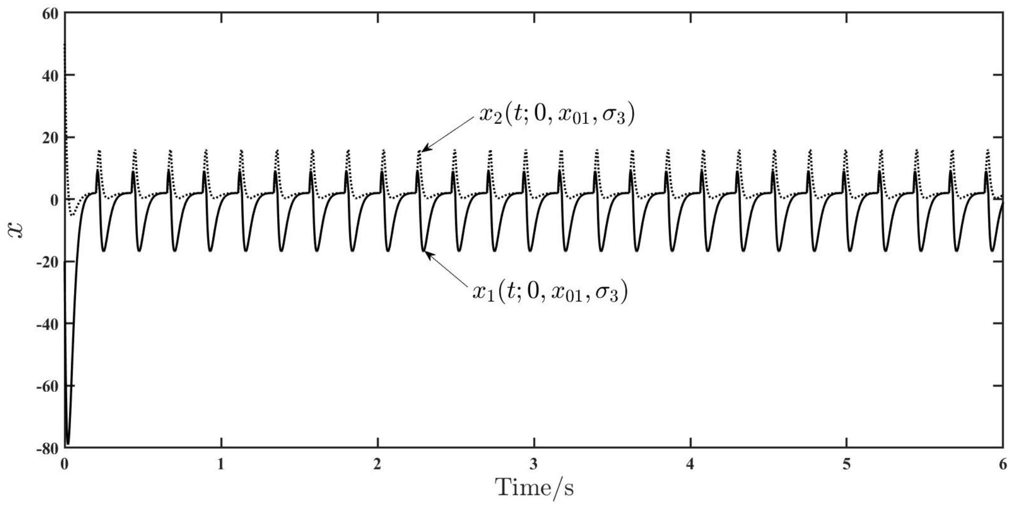

Figure 14. The trajectory of system (104) under the switching path σ_3 in relation to the switching line l_1 starting from the initial state x_{01} with respect to time t.

It is obvious from (108), (111), and (112) that $J_1 \neq 0$, $J_2 \neq 0$, $R_1 > 0$, $R_2 > 0$, and (41) are all satisfied for system (104) under the switching line l_1 denoted in (105). That is, all the conditions of Conclusion (*ii*) of Theorem 2 are satisfied for system (104). By Theorem 2, we know that system (104) under the switching line l_1 in (105) is asymptotically region stable with respect to the following region:

$$\begin{aligned}\Psi &= \left\{z \in \mathbb{R}^2 : H_1(z) \leqslant \max_{i=1,2}\left\{H_i(x^{e2})\right\}\right\} \bigcup \left\{z \in \mathbb{R}^2 : H_2(z) \leqslant \max_{i=1,2}\left\{H_i(x^{e1})\right\}\right\} \\ &= \left\{z \in \mathbb{R}^2 : H_1(z) \leqslant 750\right\} \bigcup \left\{z \in \mathbb{R}^2 : H_2(z) \leqslant 980\right\}. \end{aligned} \tag{113}$$

To show the above conclusion by simulations, we choose the following two initial states: $x_{01} = [-20\ 50]^T$ and $x_{02} = [2\ 20]^T$. It is easy to see that the two initial states x_{01} and x_{02} are contained in the interior and the

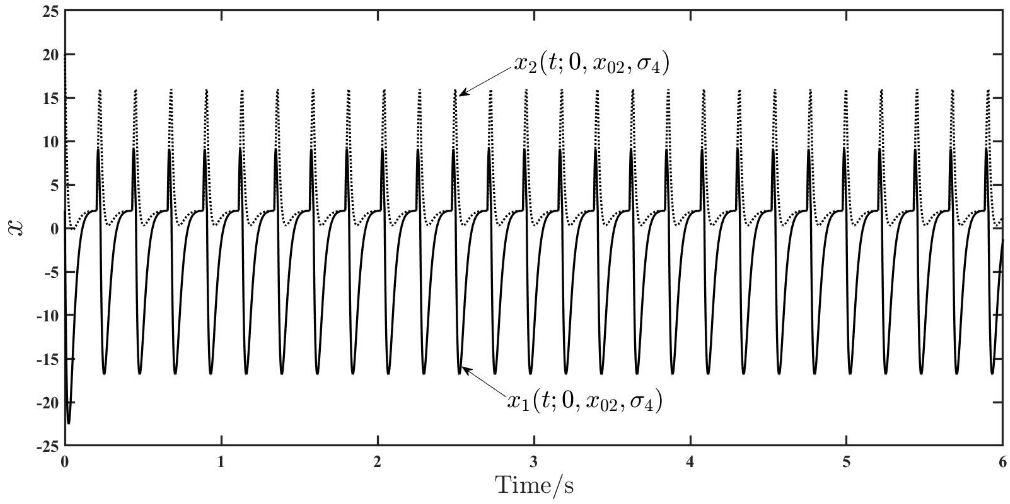

Figure 15. The trajectory of system (104) under the switching path σ_4 in relation to the switching line l_1 starting from the initial state x_{02} with respect to time t.

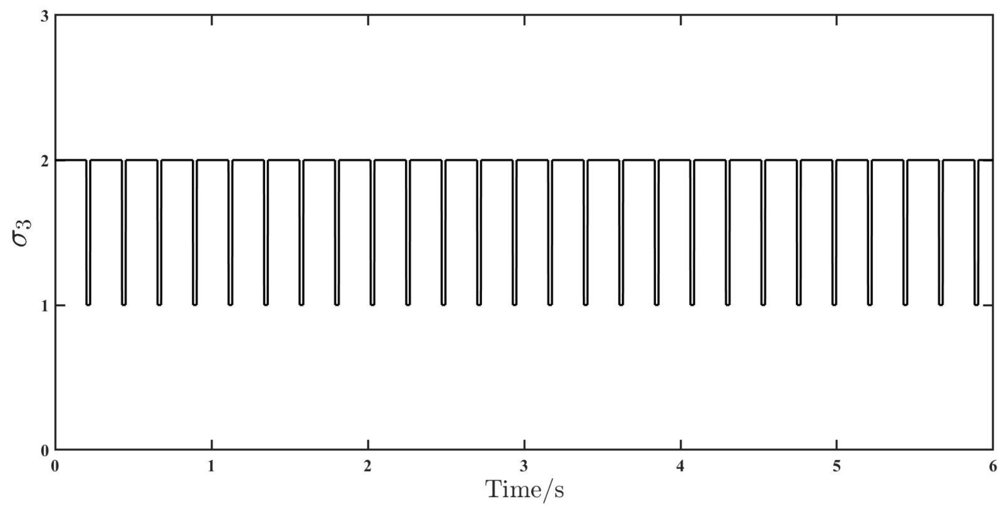

Figure 16. The response of the switching path σ_3 in relation to the switching line l_1 with respect to time t.

exterior of the region Ψ in (113), respectively. The numerical simulations are then carried out, and the results are shown in Figures 12-17, which denote the trajectories of system (104) under the switching line l_1 in (105) starting from the two initial states in the plane $\mathbb{R}^2$, the trajectories with respect to time t, and switching path with respect to time t, respectively. It can be seen from Figures 12 and 13 that the trajectories $x(t)$ of system (104) goes to/into the region Ω as $t \to +\infty$. Therefore, these simulations show that the Conclusion (ii) of Theorem 2 is effective and practical.

5. CONCLUSIONS

We have studied the region stability of two-dimensional switched linear Hamiltonian systems with multiple equilibrium points. For the case that there are two subsystems and the switching path is a switching line, by the maximum energy function method, we have proposed some sufficient conditions of region stability and

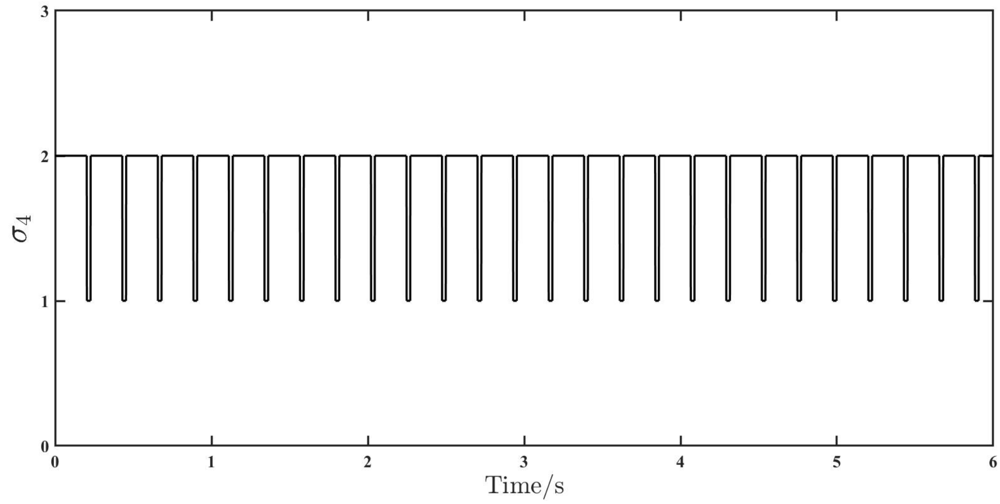

Figure 17. The response of the switching path σ_4 in relation to the switching line l_1 with respect to time t.

asymptotic region stability of such kind of switched systems. The stability criteria given are easily-test. An application of switching DC electric circuits and two numerical examples have illustrated the effectiveness and practicality of the two theorems obtained in this paper. The limitations of the stability results obtained in this paper are the following two: (1) Switched linear Hamiltonian systems with multiple equilibrium points are two-dimensional. (2) The special switching paths of switching lines. To remove the above limitations, the investigation of region stability of high-dimensional SHSs with ME under arbitrary switching paths will be our next work.

DECLARATIONS

Authors' contributions

Made substantial contributions to supervision, writing, review, editing, and methodology: Zhu L
Performed writing-original draft, software, validation, and visualization: Liu T

Availability of data and materials

Not applicable.

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Conflicts of interest

All authors declared that there are no conflicts of interest.

Ethical approval and consent to participate

Not applicable.

Consent for publication

Not applicable.

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